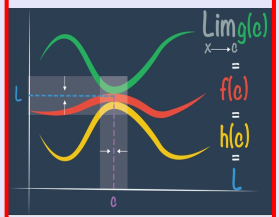
Calculus I

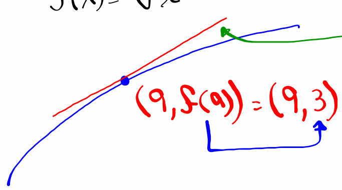
Lecture 14



Feb 19-8:47 AM

Class QZ 7

Find $\boxed{\text{slope of the tan. line}}$ to the graph of $f(x) = \sqrt{x}$ at $x = 9$.



$$m = \lim_{x \to 9} \frac{f(x) - f(9)}{x - 9}$$

$$= \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

$$= \lim_{x \to 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} = \lim_{x \to 9} \frac{\cancel{x - 9}}{\cancel{(x - 9)}(\sqrt{x} + 3)} = \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \boxed{\frac{1}{6}}$$

Eqn of tan. line at $x = a$

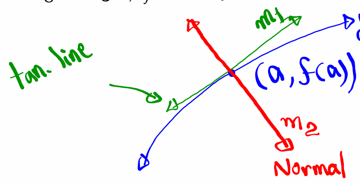
$$y - f(a) = m(x - a)$$

$$y - f(9) = \frac{1}{6}(x - 9)$$

$$y - 3 = \frac{1}{6}(x - 9) \rightarrow y = \frac{1}{6}x - \frac{9}{6} + 3 \quad \boxed{y = \frac{1}{6}x + \frac{3}{2}}$$

Feb 27-9:41 AM

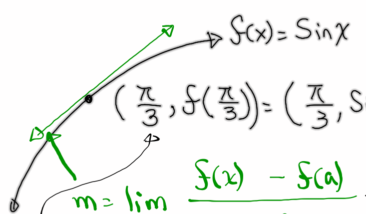
Find slope of the $\boxed{\text{normal line}}$ to the graph of $f(x) = \sin x$ at $x = \frac{\pi}{3}$.



Normal line is Perpendicular to tan. line at tangent Point.

$$m_1 \cdot m_2 = -1 \Rightarrow m_2 = \frac{-1}{m_1}$$

$$\boxed{\text{Slope of normal line} = \frac{-1}{\text{Slope of tan. line}}}$$



$\sin 60^\circ = \frac{\sqrt{3}}{2}$

$$\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = \left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{\underbrace{x - a}_{h = x - a}} = \lim_{x \to \frac{\pi}{3}} \frac{\sin x - \sin\frac{\pi}{3}}{x - \frac{\pi}{3}}$$

Feb 28-8:54 AM

Let $h = x - \frac{\pi}{3}$ as $x \to \frac{\pi}{3}$, $h \to 0$

$x = h + \frac{\pi}{3}$

$$\lim_{x \to \frac{\pi}{3}} \frac{\sin x - \sin\frac{\pi}{3}}{x - \frac{\pi}{3}} = \lim_{h \to 0} \frac{\sin\left(h + \frac{\pi}{3}\right) - \sin\frac{\pi}{3}}{h}$$

$$= \lim_{h \to 0} \frac{\sin h \cos\frac{\pi}{3} + \cos h \sin\frac{\pi}{3} - \sin\frac{\pi}{3}}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin h \cos\frac{\pi}{3}}{h} + \frac{\sin\frac{\pi}{3}\left[\cos h - 1\right]}{h} \right]$$

$$= \lim_{h \to 0} \frac{\sin h \cos\frac{\pi}{3}}{h} + \lim_{h \to 0} \frac{\sin\frac{\pi}{3}\left[\cos h - 1\right]}{h}$$

$$= \cos\frac{\pi}{3} \cdot \lim_{h \to 0} \frac{\sin h}{h} \xrightarrow{} 1 + \sin\frac{\pi}{3} \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} \xrightarrow{} 0$$

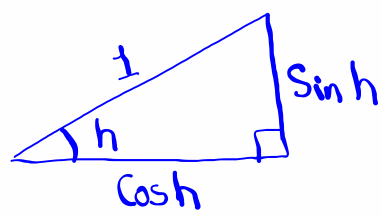
$$= \cos\frac{\pi}{3} \cdot 1 + \cancel{\sin\frac{\pi}{3} \cdot 0} \to 0 = \cos\frac{\pi}{3} = \boxed{\frac{1}{2}}$$

Slope of tan. line at $x = \pi/3$

Slope of normal line at $x = \frac{\pi}{3}$ $= \frac{-1}{1/2} = \boxed{-2}$

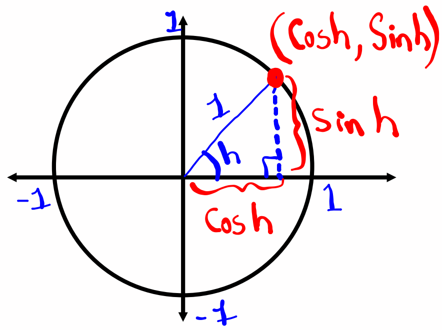
Feb 28-9:03 AM

Consider the right-triangle below



$$\sin h = \frac{\text{opposite}}{\text{hyp.}} = \frac{\text{opposite}}{1}$$

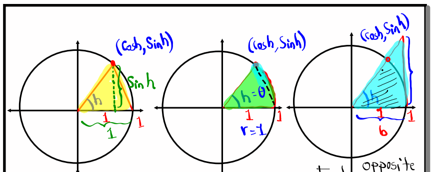
$$\sin h = \text{opposite}$$



$$\cos h = \frac{\text{adjacent}}{\text{hyp}} = \frac{\text{adjacent}}{1}$$

$$\cos h = \text{adjacent}$$

Feb 28-9:12 AM



$$\tan h = \frac{\text{opposite}}{\text{adj.}}$$

$$\tan h = \text{opposite}$$

Yellow < Green < Blue

Triangle	Sector	Triangle
$\frac{bh}{2} = \frac{1 \cdot \sin h}{2}$	$\frac{1}{2} r^2 \theta$	$\frac{bh}{2} = \frac{1 \cdot \tan h}{2}$

$$\frac{\sin h}{2} < \frac{1}{2} h < \frac{\tan h}{2}$$

multiply by 2

$$\sin h < h < \tan h$$

$$\sin h < h < \frac{\sin h}{\cos h}$$

Divide by Sin h

$$\frac{\sin h}{\sin h} < \frac{h}{\sin h} < \frac{\sin h}{\cos h \sin h}$$

$$1 < \frac{h}{\sin h} < \frac{1}{\cos h}$$

Do reciprocal $\quad 1 > \frac{\sin h}{h} > \cos h$

$$\cos h < \frac{\sin h}{h} < 1$$

$$\lim_{h \to 0} \cos h = \cos 0 = 1$$

$$\lim_{h \to 0} 1 = 1$$

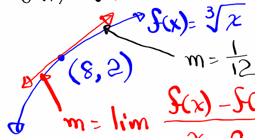
By S. T. ,

$$\boxed{\lim_{h \to 0} \frac{\sin h}{h} = \boxed{1}}$$

Feb 27-8:55 AM

Find slope of the tan. line to the graph of

$f(x)=\sqrt[3]{x}$ at $x=8$.



$f(8)=\sqrt[3]{8}=2$

$$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}\cdot\frac{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}$$

Factor $x-8$

$$x-8=(\sqrt[3]{x})^3-2^3=\boxed{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}$$

Recall

$$A^3-B^3=(A-B)(A^2+AB+B^2)$$

$$m=\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}=\lim_{x\to 8}\frac{\cancel{\sqrt[3]{x}-2}}{\boxed{\cancel{(\sqrt[3]{x}-2)}(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}}$$

$$=\lim_{x\to 8}\frac{1}{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}=\frac{1}{\sqrt[3]{8^2}+2\sqrt[3]{8}+4}$$

$$=\frac{1}{4+4+4}=\boxed{\frac{1}{12}}$$

Feb 28-9:30 AM

Find K such that

$$f(x)=\begin{cases}\dfrac{x^4-16}{x^2-4} & \text{if } x\neq 2\\ Kx & \text{if } x=2\end{cases}$$

is cont. at $x=2$.

To be cont. at $x=a$

$$\lim_{x\to a} f(x)=f(a)$$

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^4-16}{x^2-4}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x^2+4)\cancel{(x^2-4)}}{\cancel{x^2-4}}$$

$$=\lim_{x\to 2}(x^2+4)=8$$

$f(2)=K(2)=2K$

$2K=8$

$\boxed{K=4}$

Feb 28-9:47 AM

Use ε and δ to prove $\lim_{x\to 3}(x^2+5)=14$ ✓

For $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|x^2+5-14|<\varepsilon$ " $|x-3|<\delta$

$|x^2-9|<\varepsilon$ " $|x-3|<\delta$

$|x+3||x-3|<\varepsilon$ " $|x-3|<\delta$

C Keep

$|x+3|=C \to |x-3|<\frac{\varepsilon}{C}$ $\delta=\frac{\varepsilon}{C}$

If we let $\delta\le 1$

$|x-3|<1$

$-1<x-3<1$

Add 6

$5<x+3<7$ $\to |x+3|<7$

$$\boxed{\delta=\min\left\{1,\frac{\varepsilon}{7}\right\}}$$

Feb 28-9:52 AM