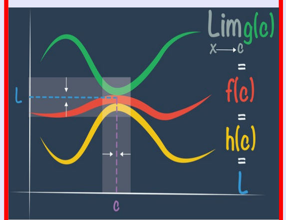
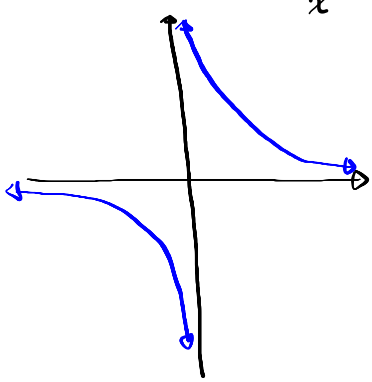
Math 261

Spring 2022

Lecture 6



Consider $f(x) = \frac{1}{x}$



$\lim_{x \to 0^+} \frac{1}{x} = \infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0} \frac{1}{x}$ D.N.E.

when $x \to 0^+$

Let $x = .001$

$f(.001) = \frac{1}{.001} = 1000$

when $x \to 0^-$

Let $x = -.001$

$f(-.001) = -1000$

$\lim_{x \to \infty} \frac{1}{x} = 0$, $\lim_{x \to -\infty} \frac{1}{x} = 0$

$x \to \infty$

Let $x = 1000$

then $\frac{1}{1000} = .001$

$\frac{1}{x} \to 0$

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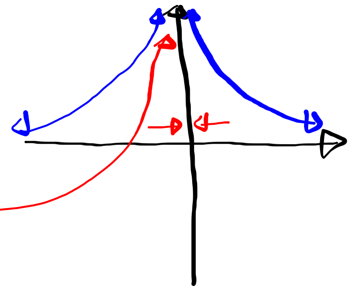
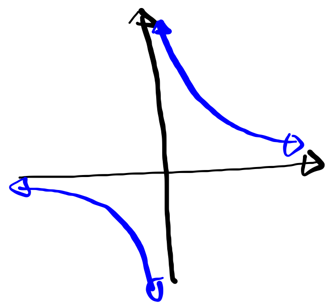
Assume $n > 0$, Integer

$f(x) = \frac{1}{x^n}$, for even integer

$\lim_{x \to 0} \frac{1}{x^n} = \infty$



for odd integer



Evaluate limits as $x \to \infty$ or $x \to -\infty$

ex: $\lim_{x \to \infty} \frac{2x-5}{6x+7}$

Divide everything by the highest power of x

$$= \lim_{x \to \infty} \frac{2x-5}{6x+7} = \lim_{x \to \infty} \frac{\frac{2x-5}{x}}{\frac{6x+7}{x}} = \lim_{x \to \infty} \frac{\frac{2x}{x} - \frac{5}{x}}{\frac{6x}{x} + \frac{7}{x}}$$

$$= \lim_{x \to \infty} \frac{2 - \frac{5}{x}}{6 + \frac{7}{x}} = \frac{\lim_{x \to \infty} 2 - 5\cancel{\lim_{x \to \infty} \frac{1}{x}}^{\to 0}}{\lim_{x \to \infty} 6 + 7\cancel{\lim_{x \to \infty} \frac{1}{x}}^{\to 0}}$$

$$= \frac{2 - 5 \cdot 0}{6 + 7 \cdot 0} = \frac{2}{6} = \boxed{\frac{1}{3}} = .\overline{3}$$

Let's say $x = 1000$

$$\frac{2(1000) - 5}{6(1000) + 7} = \frac{1995}{6007}$$

$\approx .33211...$

Evaluate $\lim\limits_{x\to-\infty} \dfrac{5x^2+x}{2x^3-7}$ Divide by highest exponent of x.

$$\lim_{x\to-\infty} \frac{5x^2+x}{2x^3-7} = \lim_{x\to-\infty} \frac{\frac{5x^2+x}{x^3}}{\frac{2x^3-7}{x^3}} = \lim_{x\to-\infty} \frac{\frac{5x^2}{x^3}+\frac{x}{x^3}}{\frac{2x^3}{x^3}-\frac{7}{x^3}}$$

$$= \lim_{x\to-\infty} \frac{\frac{5}{x}+\frac{1}{x^2}}{2-\frac{7}{x^3}} = \frac{5\lim\limits_{x\to-\infty}\frac{1}{x} \to 0 + \lim\limits_{x\to-\infty}\frac{1}{x^2} \to 0}{\lim\limits_{x\to-\infty} 2 \to 2 - 7\lim\limits_{x\to-\infty}\frac{1}{x^3} \to 0}$$

$$= \frac{5\cdot 0+0}{2-7\cdot 0} = \frac{0}{2} = \boxed{0}$$

Solve $\Rightarrow \{\ \ \}$ Solution Set

$x \in \{\ \ \}$

is an element of ✓

$x = \{\ ,\ \}$

wrong

Evaluate $\lim\limits_{x\to\infty} \dfrac{2x^2-5x}{x+3}$

Divide everything by highest exponent of x.

$$\lim_{x\to\infty} \frac{\frac{2x^2-5x}{x^2}}{\frac{x+3}{x^2}} = \lim_{x\to\infty} \frac{2-\frac{5}{x}}{\frac{1}{x}+\frac{3}{x^2}} = \frac{\lim\limits_{x\to\infty} 2 \to 2 - 5\lim\limits_{x\to\infty}\frac{1}{x} \to 0}{\lim\limits_{x\to\infty}\frac{1}{x} \to 0 + 3\lim\limits_{x\to\infty}\frac{1}{x^2} \to 0} = \frac{2}{0^+} = \boxed{+\infty}$$

Verify

let $x=100$

$$\frac{2(100)^2-5(100)}{100+3} = \frac{19500}{103} = 189.32$$

Let $x=1000$

$$\frac{2(1000)^2-5(1000)}{1000+3} = \frac{1995000}{1003} = 1989.\ldots$$

as $x\to\infty$, $\dfrac{2x^2-5x}{x+3} \to \infty$

Now limits with radicals

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{4x+5}$$

we divide everything by highest power of x.

$\sqrt{x^2+1} \approx \sqrt{x^2} = x$ as $x\to\infty$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{4x+5} = \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{x}}{\frac{4x+5}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}{4+\frac{5}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+1}{x^2}}}{4+\frac{5}{x}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{4+\frac{5}{x}}$$

$$= \frac{\sqrt{\lim\limits_{x\to\infty} \cancel{1}^{\,1} + \lim\limits_{x\to\infty} \cancel{\frac{1}{x^2}}^{\,0}}}{\lim\limits_{x\to\infty} \cancel{4}^{\,4} + 5\lim\limits_{x\to\infty} \cancel{\frac{1}{x}}^{\,0}} = \frac{\sqrt{1}}{4} = \boxed{\frac{1}{4}} = .25$$

check

$x = 100$ $\quad \frac{\sqrt{(100)^2+1}}{4(100)+5} = .2469$

$\to \approx .25$

$x = 1000$ $\quad \frac{\sqrt{(1000)^2+1}}{4(1000)+5} = .2496$

Evaluate $\lim\limits_{x\to-\infty} \frac{\sqrt{x^2-2}}{4x-5}$

$\sqrt{x^2-2} \approx \sqrt{x^2} = -x$

$x\to-\infty$

$$\sqrt{x^2} = |x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x\geq 0 \end{cases}$$

$\sqrt{x^2} = -x$

$-\sqrt{x^2} = x$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-2}}{4x-5} = \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2-2}}{x}}{\frac{4x-5}{x}}$$

$$= \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2-2}}{-\sqrt{x^2}}}{4-\frac{5}{x}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{x^2-2}{x^2}}}{4-\frac{5}{x}} = -\frac{\sqrt{\lim\limits_{x\to-\infty} 1 - \lim\limits_{x\to-\infty} \cancel{\frac{2}{x^2}}^{\,0}}}{\lim\limits_{x\to-\infty} 4 - \lim\limits_{x\to-\infty} \cancel{\frac{5}{x}}^{\,0}}$$

$$= -\frac{\sqrt{1}}{4} = \boxed{\frac{-1}{4}}$$

Evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = \infty-\infty$ I.F.

$\frac{0}{0}$ $\frac{\infty}{\infty}$

Suppose $x=100$

$\sqrt{100^2+1}-100 = .005$

what about $x=1000$

$\sqrt{1000^2+1}-1000=.0005$

as $x\to\infty$, $\sqrt{x^2+1}-x \to 0$

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}-x\right)}{1}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

Divide by x,

Recall

$x=\sqrt{x^2}$

$x\to\infty$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{\frac{x^2+1}{x^2}}+\frac{x}{x}}=\frac{0}{1+1}$$

$$=\frac{0}{2}=\boxed{0}$$

Evaluate

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right)=\infty-\infty \quad \text{I.F.}$$

$$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+4x}-x}{1}\cdot\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x}\right]$$

$$=\lim_{x\to\infty}\frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x}=\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+4x}+x}$$

Recall

as $x\to\infty$

$x=\sqrt{x^2}$

$$=\lim_{x\to\infty}\frac{\frac{4x}{x}}{\sqrt{1+\frac{4}{x}}+\frac{x}{x}}$$

$$=\lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{4}{x}}+1}=\frac{4}{1+1}=\boxed{2}$$

Evaluate

1) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{h\to 0} \sin h$

Let $h = \frac{1}{x}$

as $x \to \infty$

$h \to 0$

$= \sin 0$

$= \boxed{0}$

2) $\lim_{x\to\infty} x \sin\frac{1}{x}$

Let $h = \frac{1}{x}$, $x = \frac{1}{h}$

as $x \to \infty$

$h \to 0$

$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{h\to 0} \frac{1}{h}\sin h$

$= \lim_{h\to 0} \frac{\sin h}{h} = \boxed{1}$

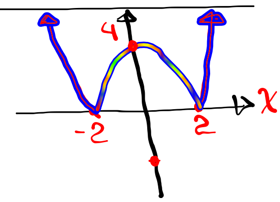
Make Sure You are in radian mode,

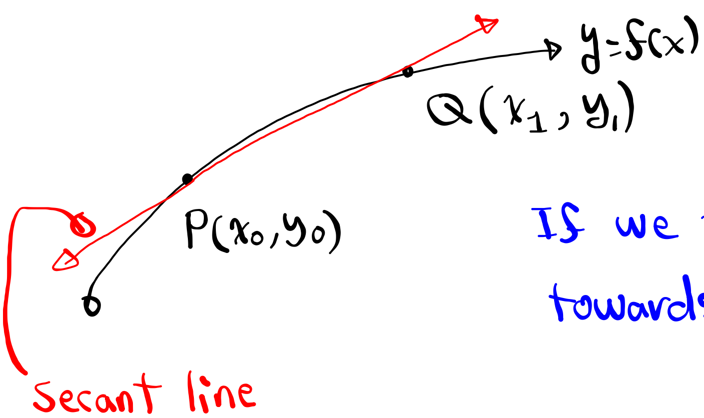
Let $x = 100$

$100 \sin\frac{1}{100} = .9999\ldots \approx 1$

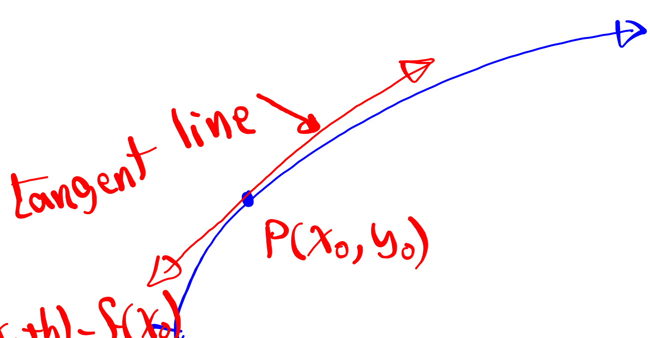
Online QZ 3

Graph $f(x) = |x^2 - 4|$





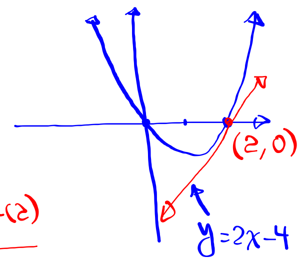
If we move point Q towards point P



$$m_{\text{tan. line}} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Find equation of the tan. line at $x=2$

for $f(x)=x^2-2x$.



$$f(2)=2^2-2(2)$$
$$=4-4$$
$$=0$$

$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2-2(2+h)-0}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4-2h}{h}$$

$$=\lim_{h\to 0}\frac{h^2+2h}{h}=\lim_{h\to 0}\frac{h(h+2)}{h}=\lim_{h\to 0}(h+2)$$

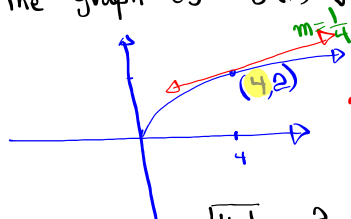
$$\Rightarrow=\boxed{2}$$

Using point-slope formula

$$y-y_1=m(x-x_1)$$
$$y-0=2(x-2)\Rightarrow\boxed{y=2x-4}$$

Find equation of the tangent line to the graph of $f(x)=\sqrt{x}$ at $x=4$.

$$f(4)=\sqrt{4}=2$$



$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$

$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}=\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$$=\lim_{h\to 0}\frac{\cancel{4}+h-\cancel{4}}{h(\sqrt{4+h}+2)}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}=\frac{1}{\sqrt{4+0}+2}=\boxed{\frac{1}{4}}$$

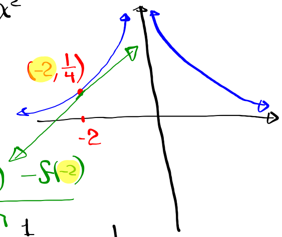
Use point-slope formula

$$y-y_1=m(x-x_1)$$
$$y-2=\frac{1}{4}(x-4)\Rightarrow\boxed{y=\frac{1}{4}x+1}$$

For wednesday $\Rightarrow$ Google: Normal Line

Find equation of the tan. line to the graph of $f(x)=\frac{1}{x^2}$ at the point where $x=-2$.

$f(-2)=\frac{1}{(-2)^2}=\frac{1}{4}$



$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{(-2+h)^2}-\frac{1}{4}}{h}$$ LCD $=4(-2+h)^2$

$$=\lim_{h\to 0}\frac{4-(-2+h)^2}{4h(-2+h)^2}=\lim_{h\to 0}\frac{\cancel{4}\cancel{-4}+4h-h^2}{4h(-2+h)^2}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(4-h)}{4\cancel{h}(-2+h)^2}$$

$$=\lim_{h\to 0}\frac{4-h}{4(-2+h)^2}=\frac{4}{4(-2)^2}=\boxed{\frac{1}{4}}$$

Using Point-Slope

$y-y_1=m(x-x_1)$

$y-\frac{1}{4}=\frac{1}{4}(x-{-2})$

LCD $=4$

$4y-1=1(x+2)$

$4y-1=x+2$

$4y=x+3$

$\boxed{y=\frac{1}{4}x+\frac{3}{4}}$

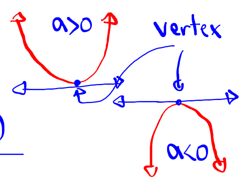
tan. line at $(-2,\frac{1}{4})$ for $f(x)=\frac{1}{x^2}$.

Find points with horizontal tan. line for any quadratic function.

$m=0$

$f(x)=ax^2+bx+c$

$a\neq 0$



$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{a(x+h)^2+b(x+h)+c-ax^2-bx-c}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{ax^2}+2axh+ah^2\cancel{+bx}+bh\cancel{+c}\cancel{-ax^2}\cancel{-bx}\cancel{-c}}{h}$$

$$=\lim_{h\to 0}\frac{2axh+ah^2+bh}{h}=\lim_{h\to 0}\frac{\cancel{h}(2ax+ah+b)}{\cancel{h}}$$

$$=\lim_{h\to 0}(2ax+\underbrace{ah}_{\to 0}+b)=2ax+b$$

we want $m=0 \Rightarrow 2ax+b=0 \Rightarrow x=\frac{-b}{2a}$

Final Ans: $\boxed{\left(\frac{-b}{2a},\ f\left(\frac{-b}{2a}\right)\right)}$

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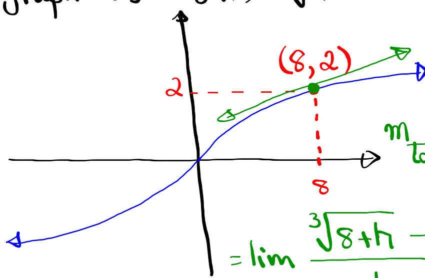
Find slope of the tan. line to the graph of $f(x)=\sqrt[3]{x}$ at $x=8$.

$f(8)=\sqrt[3]{8}=2$



$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(8+h)-f(8)}{h}$

$=\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}$

Using A^3-B^3 concept.

$=\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}\cdot\frac{\sqrt[3]{(8+h)^2}+2\sqrt[3]{8+h}+4}{\sqrt[3]{(8+h)^2}+2\sqrt[3]{8+h}+4}$

$=\lim_{h\to 0}\frac{\cancel{8}+\cancel{h}^{1}\cancel{-8}}{\cancel{h}\left[\sqrt[3]{(8+h)^2}+2\sqrt[3]{8+h}+4\right]}$

$=\lim_{h\to 0}\frac{1}{\sqrt[3]{(8+h)^2}+2\sqrt[3]{8+h}+4}=\frac{1}{4+4+4}=\boxed{\frac{1}{12}}$

Class QZ 3

Prove $\lim_{x\to 2}(3x+5)=11$.

$f(x)=3x+5$

$L=11$

$a=2$

$\lim_{x\to 2}(3x+5)=3(2)+5=11$ ✓

For $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|3x+5-11|<\varepsilon$ " $|x-2|<\delta$

$|3x-6|<\varepsilon$ " $|x-2|<\delta$

$|3(x-2)|<\varepsilon$ " $|x-2|<\delta$

$3|x-2|<\varepsilon$ " $|x-2|<\delta$

$|x-2|<\frac{\varepsilon}{3}$ So Pick $\boxed{\delta=\frac{\varepsilon}{3}}$