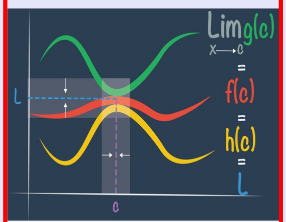
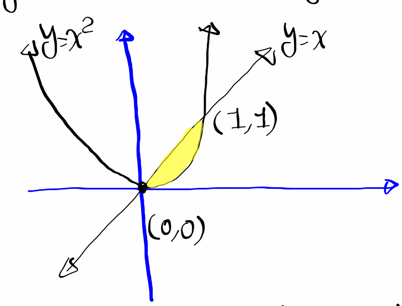
Math 261
Spring 2022
Lecture 29



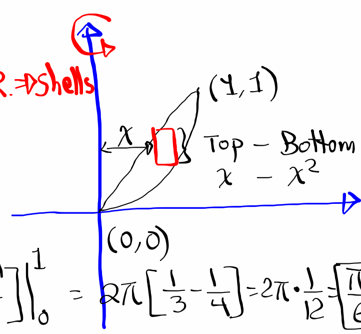
1) Draw the enclosed region R bounded by $y = x$ and $y = x^2$.



2) Find the volume by rotating region R by Y-axis

Cross-Section ⊥ x-axis ⟹ dx

Cross-Section is parallel to A.O.R. ⟹ Shells



$$V = \int_0^1 2\pi(x)\cdot(x - x^2)\,dx$$

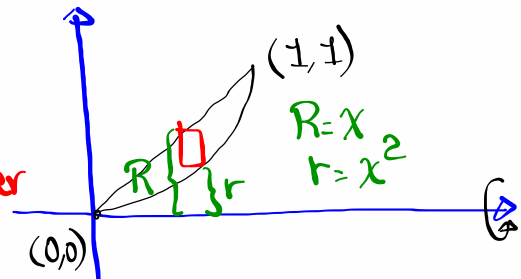
$$= 2\pi\int_0^1 (x^2 - x^3)\,dx = 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]\Big|_0^1 = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = 2\pi\cdot\frac{1}{12} = \boxed{\frac{\pi}{6}}$$

3) Find the volume by rotating region R by the x-axis

Cross-Section $\perp$ x-axis $\Rightarrow dx$

Cross-Section $\perp$ A.O.R.

R is not totally attached to A.O.R. $\Rightarrow$ washer



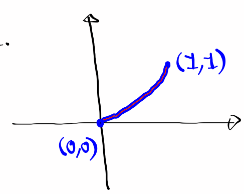
$$V = \int_0^1 \pi\left[(x)^2 - (x^2)^2\right]dx = \pi\int_0^1 \left[x^2 - x^4\right]dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]\Bigg|_0^1$$

$$= \pi\left[\frac{1}{3} - \frac{1}{5}\right] = \pi \cdot \frac{2}{15} = \boxed{\frac{2\pi}{15}}$$

Arc length (Calc. II)

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \quad \text{or} \quad L = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$$

Find the arc length of $f(x) = x\sqrt{x}$ from $x=0$ to $x=1$.



$$L = \int_0^1 \sqrt{1 + [f'(x)]^2}\,dx$$

$f(x) = x\sqrt{x}$

$f(x) = x^{3/2}$ $\quad f'(x) = \frac{3}{2}x^{1/2}$

$1 + [f'(x)]^2 = 1 + \frac{9}{4}x$

$$L = \int_0^1 \sqrt{1 + \tfrac{9}{4}x}\,dx$$

$u = 1 + \frac{9}{4}x$ $\quad du = \frac{9}{4}dx$ $\quad \frac{4}{9}du = dx$

$x=0 \rightarrow u=1$, $x=1 \rightarrow u = \frac{13}{4}$

$$= \int_1^{13/4} \sqrt{u} \cdot \frac{4}{9}du = \frac{4}{9} \cdot \frac{u^{3/2}}{3/2}\Bigg|_1^{13/4}$$

$$= \frac{8}{27}\left[u\sqrt{u}\right]\Big|_1^{13/4}$$

$$= \frac{8}{27}\left[\frac{13}{4} \cdot \sqrt{\frac{13}{4}} - 1\sqrt{1}\right] = \frac{8}{27}\left[\frac{13\sqrt{13}}{8} - 1\right]$$

$$= \frac{\cancel{8}}{27}\left[\frac{13\sqrt{13} - 8}{\cancel{8}}\right]$$

$$\boxed{\frac{13\sqrt{13} - 8}{27}}$$

Surface Area of Revolution

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx \quad \text{by } x\text{-axis}$$

$$S = \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy \quad \text{by } Y\text{-axis.}$$

Find the surface area by rotating the curve $f(x) = \sqrt{1-x^2}$ for $-\frac{1}{2} \le x \le \frac{1}{2}$ about x-axis.

First $f(x) = \sqrt{1-x^2}$

$y = \sqrt{1-x^2}$

$f(x) = (1-x^2)^{1/2}$

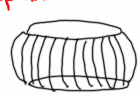
$f'(x) = \dfrac{1 \cdot -2x}{2(1-x^2)^{1/2}} = \dfrac{-x}{\sqrt{1-x^2}}$

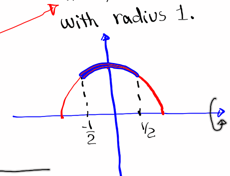
Square both sides $y^2 = 1 - x^2$

$x^2 + y^2 = 1$ Circle, center at $(0,0)$ with radius 1.

$f(x) \ge 0$

Top half of





$$S = \int_{-1/2}^{1/2} 2\pi f(x) \cdot \sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_{-1/2}^{1/2} 2\pi \cdot \sqrt{1-x^2} \cdot \sqrt{1 + \left[\frac{-x}{\sqrt{1-x^2}}\right]^2}\,dx$$

$$= \int_{-1/2}^{1/2} 2\pi \sqrt{1-x^2} \cdot \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}}\,dx$$

$$= \int_{-1/2}^{1/2} 2\pi \sqrt{1-x^2} \cdot \sqrt{\frac{1}{1-x^2}}\,dx$$

$$= \int_{-1/2}^{1/2} 2\pi \cancel{\sqrt{1-x^2}} \cdot \frac{1}{\cancel{\sqrt{1-x^2}}}\,dx = \int_{-1/2}^{1/2} 2\pi\,dx$$

$$= 2\pi \cdot 2\int_0^{1/2} dx = 4\pi \cdot x\Big|_0^{1/2} = 4\pi \cdot \frac{1}{2} = \boxed{2\pi}$$

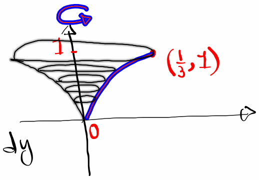
$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ when $f(x)$ is $\boxed{\text{even}}$ $f(x)=1$, $f(-x)=1$

Find the surface area generated by revolving $y=\sqrt[3]{3x}$ about the y-axis for $0\le y\le 1$

$y^3=3x \to x=\frac{1}{3}y^3$

$x=g(y)=\frac{1}{3}y^3$

$g'(y)=y^2$



$$S=\int_0^1 2\pi\, g(y)\sqrt{1+[g'(y)]^2}\,dy$$

$$=\int_0^1 2\pi\cdot\frac{1}{3}y^3\cdot\sqrt{1+(y^2)^2}\,dy$$

$$=\frac{2\pi}{3}\int_0^1 y^3\sqrt{1+y^4}\,dy$$

$u=1+y^4$

$du=4y^3dy$

$\frac{du}{4}=y^3dy$

$y=0 \quad u=1$

$y=1 \quad u=2$

$$=\frac{2\pi}{3}\int_1^2\sqrt{u}\,\frac{du}{4}=\frac{2\pi}{3}\cdot\frac{1}{4}\cdot\frac{u^{3/2}}{3/2}\Big|_1^2$$

$$=\frac{\cancel{2}\pi}{3}\cdot\frac{1}{\cancel{4}}\cdot\frac{\cancel{2}}{3}\left[u\sqrt{u}\right]\Big|_1^2$$

$$=\frac{\pi}{9}\left[2\sqrt{2}-1\sqrt{1}\right]=\boxed{\frac{\pi(2\sqrt{2}-1)}{9}}$$

Final exam:
Monday June 6, 2022
7:00 – 9:00
Use the link for office hours to join the meeting

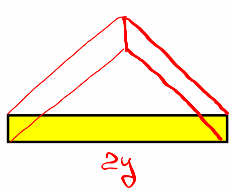
Consider the enclosed region by $x^2+y^2=1$

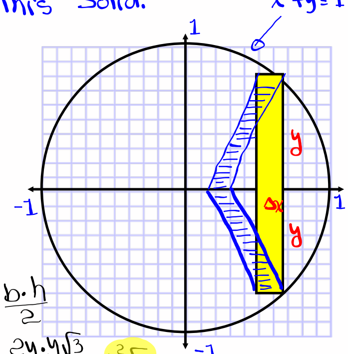
$y^2=1-x^2$

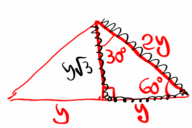
Draw cross-sections $\perp$ x-axis.

Consider a solid in the form of equilateral triangle with its base on the enclosed region.

Find the volume of this solid.







$\text{Area}=\frac{b\cdot h}{2}$

$=\frac{2y\cdot y\sqrt{3}}{2}=y^2\sqrt{3}$

$$V=\int_{-1}^{1}\text{Area of Cross-Section}\,dx=\int_{-1}^{1}y^2\sqrt{3}\,dx=\sqrt{3}\int_{-1}^{1}(1-x^2)\,dx$$

even

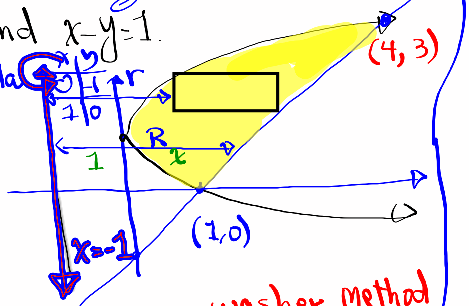
$$=\sqrt{3}\cdot 2\int_0^1(1-x^2)\,dx=2\sqrt{3}\left[x-\frac{x^3}{3}\right]\Big|_0^1=2\sqrt{3}\cdot\frac{2}{3}=\boxed{\frac{4\sqrt{3}}{3}}$$

Find the volume by rotated the enclosed region below by $x=-1$.

Answer $\frac{117\pi}{5}$

enclosed region is bounded by $x=(y-1)^2$ and $x-y=1$ line

Sideway Parabola
vertex $(0,1)$
opens right



$(y-1)^2 - y = 1$

$y^2 - 2y \cancel{+1} - y = \cancel{1}$

$y^2 - 3y = 0 \quad y=0,\ y=3$

washer method

$R = x_{line} + 1$

$r = x_{curve} + 1$

$R = y+1+1 = y+2$

$r = (y-1)^2 + 1 = y^2 - 2y + 2$

$$V = \int_0^3 \pi\left[(y+2)^2 - (y^2-2y+2)^2\right]dy =$$

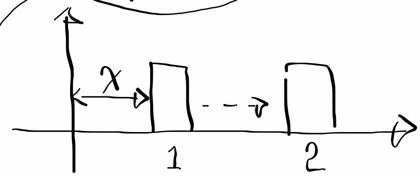
Introduction to work:

Work done in moving an object from a to b is

$$W = \int_a^b f(x)\,dx$$

$f(x)$ is the force applied to object to move from a to b.

Ex: a particle is located x feet from the origin and a force of x^3+x pounds acts on it. How much is required to move it from 1 ft to 2 ft?



$$W = \int_1^2 f(x)\,dx = \int_1^2 (x^3+x)\,dx = \left(\frac{x^4}{4}+\frac{x^2}{2}\right)\Big|_1^2$$

$$= \left(\frac{2^4}{4}+\frac{2^2}{2}\right) - \left(\frac{1^4}{4}+\frac{1^2}{2}\right)$$

$$= 4 + 2 - \frac{3}{4} = 6 - \frac{3}{4} = \frac{21}{4} = \boxed{5.25 \text{ ft/lb}}$$

Hooke's Law

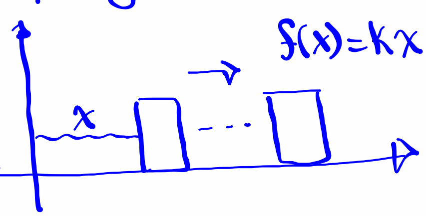
$$f(x) = kx$$

k is spring constant

$k > 0$

when x is not too large.

Hooke's law is the force applied to a spring to stretch it.



A spring has a length of 10cm.

A force of 40 N is applied to stretch it to 15cm.

displacement: 15cm to 10cm = 5cm

by Hooke's Law $f(x) = Kx$

$40 = K \cdot (.05)$ $\quad$ 5cm = .05 m

$K = \frac{40}{.05}$ $\quad$ $K = 800$

$f(x) = 800x$

How much work is required to stretch the spring from 15cm to 18cm?

10cm → 15cm: 5cm = .05m

10cm → 18cm: 8cm = .08 m

$$W = \int_a^b f(x)\,dx = \int_{.05}^{.08} 800x\,dx$$

$$= 400x^2 \Big|_{.05}^{.08} = \boxed{1.56\ J}$$

m-N ⇒ Jewls

A spring has a natural length of 24 inches.

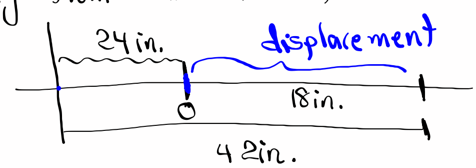
A force of 5 lb applies to stretch 10 inches beyond the natural length.

By Hooke's Law

$f(x) = Kx$ ← displacement

$5 = K \cdot 10 \Rightarrow K = \frac{1}{2} \Rightarrow f(x) = \frac{1}{2}x$

How much work is required to stretch the spring from natural length to 42 inches length.



$$W = \int_0^{18} \frac{1}{2}x\,dx = \frac{1}{4}x^2 \Big|_0^{18} = \boxed{81\ \text{in-lb}}$$

Final exam

1) Read my emails from now to final exam
2) Final Monday June 6, 2022 7:00-9:00
3) You may arrive early or stay longer for reasonable extra time.
4) You must be in the Zoom meeting no later than 7:15.
5) No emails regarding grade after final, but you can attend office hours.
6) Use office hours Zoom link to join.
7) No class nextweek → there are office hours.