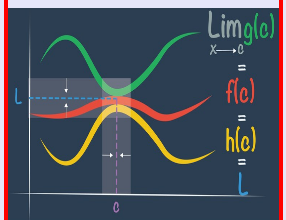
Math 261

Spring 2022

Lecture 27



Class QZ 16

Find the average value of $\underbrace{f(x) = \cos^4 x \sin x}$ on $[0, \pi]$. Exact answer only.

Continuous everywhere

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi - 0}\int_0^{\pi} \cos^4 x \sin x\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$x = 0, u = 1$
$x = \pi, u = -1$

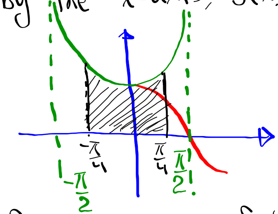
$$= \frac{1}{\pi}\int_1^{-1} u^4 \cdot (-du) = \frac{1}{\pi}\int_{-1}^{1} u^4\,du$$

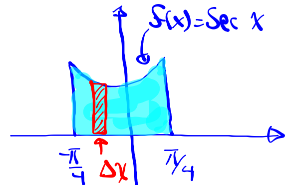
$$= \frac{2}{\pi}\int_0^1 u^4\,du = \frac{2}{\pi}\cdot\frac{u^5}{5}\Big|_0^1 = \boxed{\frac{2}{5\pi}}$$

✓

Consider $f(x) = \sec x$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

1) Draw and shade the enclosed region bounded by the x-axis, $f(x)$ on the given interval.





2) Find the area of the enclosed region from above.

$$A = \int_{-\pi/4}^{\pi/4} [\sec x - 0]\,dx = \int_{-\pi/4}^{\pi/4} \sec x\,dx = 2\int_0^{\pi/4} \sec x\,dx$$

$$= 2\left[\ln|\sec x + \tan x|\right]\Big|_0^{\pi/4}$$

Calc. II

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$= 2\left[\ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln(\sec 0 + \tan 0)\right]$$

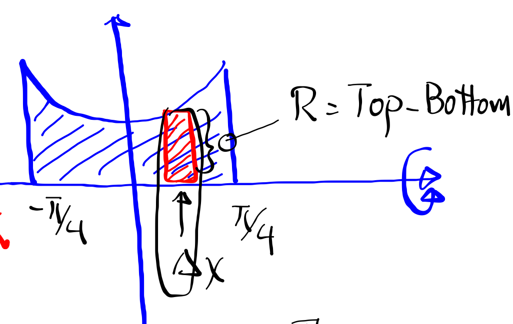
$$= 2\left[\ln(\sqrt{2} + 1) - \ln(1 + 0)\right]$$

$$= 2\left[\ln(\sqrt{2}+1) - \cancelto{0}{\ln 1}\right] = \boxed{2\ln(\sqrt{2}+1)}$$

3) Find the volume by rotating above region by x-axis.

Region is (Totally) attached to the A.O.R.

⇒ Disk



Cross-Section ⊥ to A.O.R.

$$V = \int_{-\pi/4}^{\frac{\pi}{4}} \pi R^2\,dx = \pi\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = 2\pi\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= 2\pi \cdot \tan x\Big|_0^{\pi/4} = 2\pi\left(\cancelto{1}{\tan\pi/4} - \cancelto{0}{\tan 0}\right) = 2\pi(1) = \boxed{2\pi}$$

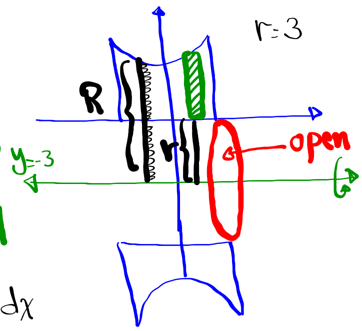
4) ~~Find~~ Set-up the volume by rotating the enclosed region from part 1 by $y=-3$.

$R = \sec x + 3$

$r = 3$

enclosed region is not totally attached to the A.O.R.

Cross-Section $\perp$ A.O.R



Washer Method

$$V = \int_{-\frac{\pi}{4}}^{\pi/4} \pi \left[R^2 - r^2 \right] dx$$

Set-up the integral.

$$= \int_{-\pi/4}^{\frac{\pi}{4}} \pi \left[(\sec x + 3)^2 - 3^2 \right] dx$$

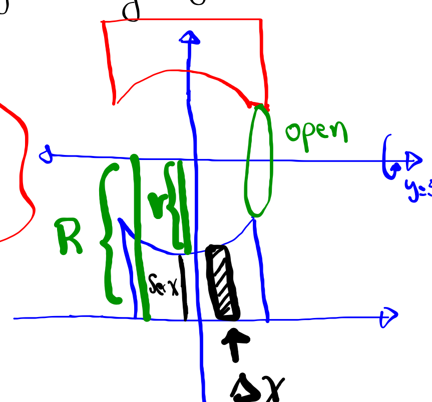
$$= \int_{-\pi/4}^{\pi/4} \pi \left[\sec^2 x + 6 \sec x \right] dx$$

To finish this, we need Calc. II.

5) Set-up the integral for the volume by rotating above Region by $y=5$.

1) Cross-Section $\perp$ A.O.R.

2) Region is Not totally attached to A.O.R.



Washer Method

$$V = \int_{-\pi/4}^{\pi/4} \pi \left[R^2 - r^2 \right] dx$$

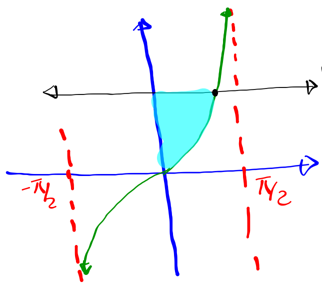
$R = 5$

$\sec x + r = 5$

$r = 5 - \sec x$

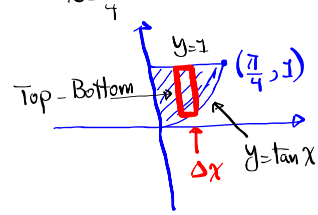
$$= \int_{-\pi/4}^{\frac{\pi}{4}} \pi \left[5^2 - (5 - \sec x)^2 \right] dx$$

1) Draw and shade the region bounded by y-axis, $y=1$, and $f(x)=\tan x$.



$\tan x = 1$

$x = \frac{\pi}{4}$



2) Find the area of this enclosed region.

$$A = \int_0^{\frac{\pi}{4}} [\text{Top} - \text{Bottom}]\,dx = \int_0^{\pi/4} [1 - \tan x]\,dx$$

Calc. II

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$= \left(x - \ln(\sec x)\right)\Big|_0^{\pi/4}$$

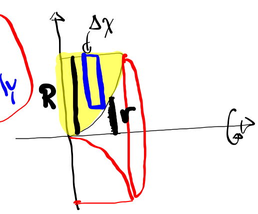
$$= \left(\frac{\pi}{4} - \ln\left(\underbrace{\sec\frac{\pi}{4}}_{\sqrt{2}}\right)\right) - \left(0 - \cancel{\ln\left(\underbrace{\sec 0}_{1}\right)}^{0}\right)$$

$$= \boxed{\frac{\pi}{4} - \ln\sqrt{2}}$$

3) Find the volume by rotating this enclosed region by the x-axis.

1) Cross-Section $\perp$ A.O.R.

2) Enclosed region not totally attached to A.O.R.



Washer Method

$R = 1$

$r = \tan x$

Recall

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$-\tan^2 x = -\sec^2 x + 1$

$$V = \int_0^{\pi/4} \pi\left[R^2 - r^2\right]dx$$

$$= \pi\int_0^{\pi/4} \left[1 - \tan^2 x\right]dx$$

$$= \pi\int_0^{\pi/4} \left[1 - \sec^2 x + 1\right]dx$$

$$= \pi\left[2x - \tan x\right]\Big|_0^{\pi/4} = \pi\left[2\cdot\frac{\pi}{4} - \tan\frac{\pi}{4} - \cancel{2(0)}^{0} + \cancel{\tan 0}^{0}\right]$$

$$= \pi\left[\frac{\pi}{2} - 1\right]$$

$$= \boxed{\frac{\pi(\pi - 2)}{2}}$$

Evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x^2 \, dx$$

$u = x^2$

$du = 2x\, dx$

$\frac{du}{2} = x\, dx$

$x = -\frac{\pi}{2} \to u = \frac{\pi^2}{4}$

$x = \frac{\pi}{2} \to u = \frac{\pi^2}{4}$

$$= \int_{\frac{\pi^2}{4}}^{\frac{\pi^2}{4}} \sin u \, \frac{du}{2} = \boxed{0}$$

$$\int_a^a f(x)dx = 0$$

find

$$\int_{-\pi/3}^{\pi/3} x^4 \sin x \, dx = \boxed{0}$$

$f(x) = x^4 \sin x$

$f(-x) = (-x)^4 \sin(-x)$

$= x^4 \cdot -\sin x$

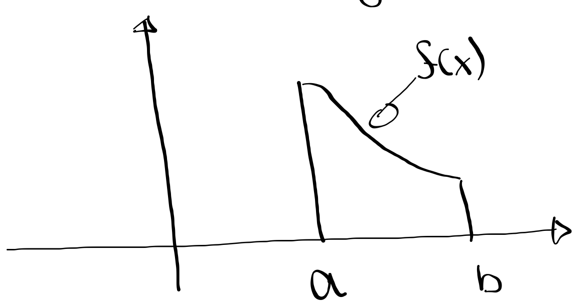
$= -x^4 \sin x$

$= -f(x)$

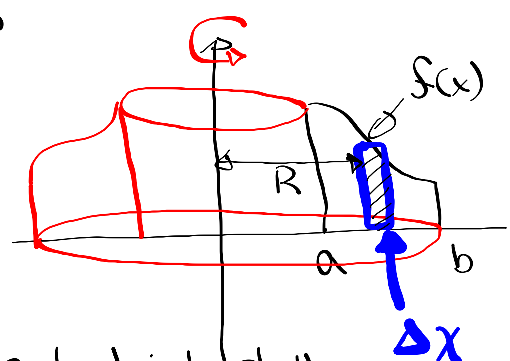
$\therefore f(x)$ is an odd function

$$\int_{-a}^{a} \text{odd function}\, dx = 0$$

Consider the region below

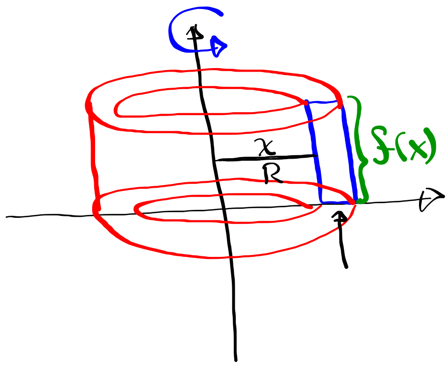


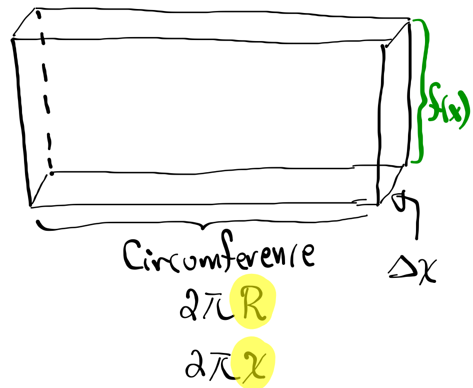
Rotate by Y-axis



Cross-Section is Parallel to A.O.R.

↳ Cylindrical Shells Method.





$$V = \int_a^b 2\pi x \cdot f(x) \cdot dx$$

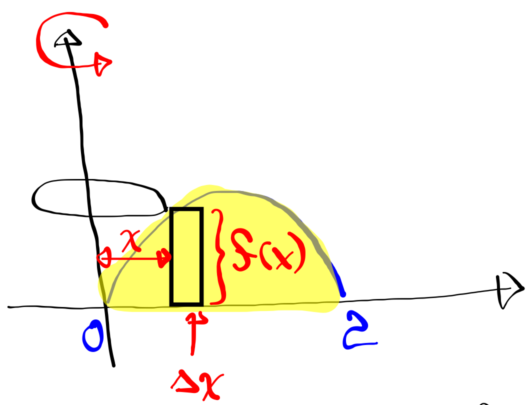
Circumference Height Thickness

Draw the region enclosed by $f(x) = 2x - x^2$ and x-axis, then rotate about y-axis.

Find the volume.

Cross-Section is Parallel to A.O.R.

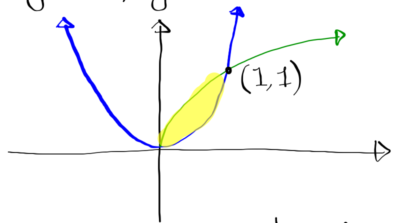
Cylindrical Shell



$$V = \int_0^2 2\pi x (2x - x^2)\, dx = 2\pi \int_0^2 (2x^2 - x^3)\, dx = 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]\Bigg|_0^2$$

$$= \boxed{}\ \checkmark$$

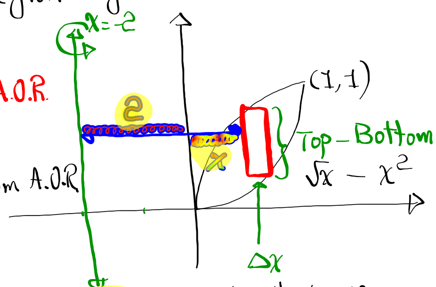
1) Draw and shade the region enclosed by $y=\sqrt{x}$, $y=x^2$ in QI.



Rotate this region by $x=-2$. Find the Volume

Cross-Section is Parallel to the A.O.R.

How far is the cross-section from A.O.R

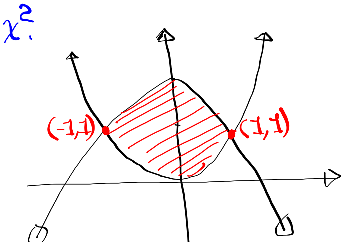


$V=\int 2\pi \cdot$ distance from A.O.R. $\cdot$ Height $\cdot$ thickness

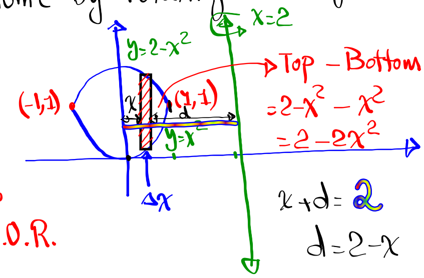
Foil, Simplify, integrate, evaluate

$=\int_0^1 2\pi(2+x)\cdot(\sqrt{x}-x^2)\,dx = \boxed{}$

1) Draw the region enclosed by $y=x^2$ and $y=2-x^2$.



2) Find the Volume by rotating this region by $x=2$.



Top – Bottom
$=2-x^2-x^2$
$=2-2x^2$

$x+d=2$
$d=2-x$

Cross-Section is Parallel to the A.O.R.

Shell Method

$V=\int_{-1}^{1} 2\pi\cdot(2-x)(2-2x^2)\,dx = 2\pi\int_{-1}^{1}(2-x)(2-2x^2)\,dx$

distance from A.O.R.

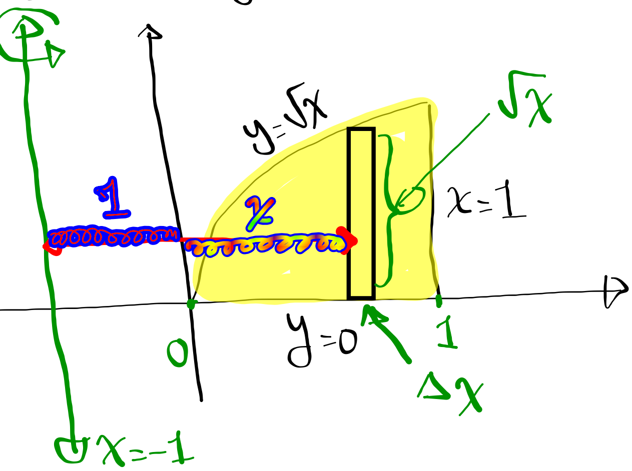
$= \boxed{}$

Find the volume by rotating the region below by $x=-1$.

Shells

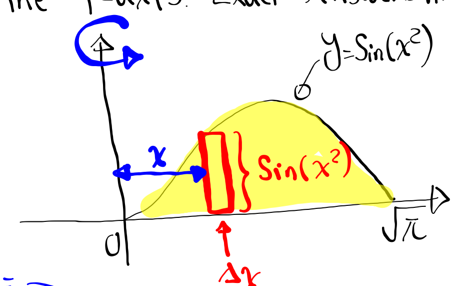
$$V=\int_0^1 2\pi(1+x)\sqrt{x}\,dx$$

distance from A.O.R.



Find the volume by rotating the enclosed region below by the Y-axis. Exact Answer only

Class QZ 17



Using Shells Method

$$V=\int_0^{\sqrt{\pi}} 2\pi x\sin x^2\,dx = \pi\int_0^{\pi}\sin u\,du$$

$u=x^2$ $\quad x=0 \rightarrow u=0$

$du=2x\,dx$ $\quad x=\sqrt{\pi} \rightarrow u=\pi$

$$=\pi\left[-\cos u\right]\Big|_0^{\pi} = -\pi\left[\cos\pi-\cos 0\right]$$

$$=-\pi\left[-1-1\right]$$

$$=\boxed{2\pi}$$