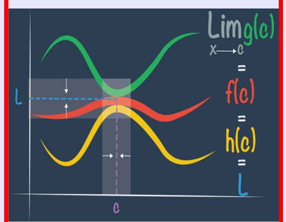
# Math 261

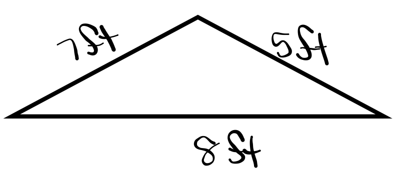
## Spring 2022

## Lecture 19




Class QZ 11

Find the exact area of the triangle below:




Heron's Formula

$A = \sqrt{S(S-a)(S-b)(S-c)}$

where $S = \frac{P}{2}$

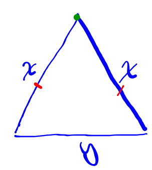
$P = a + b + c = 20$ ft

$S = \frac{P}{2} = 10$

$\text{Area} = \sqrt{10(10-8)(10-7)(10-5)} = \sqrt{10 \cdot 2 \cdot 3 \cdot 5} = \sqrt{300} = \boxed{10\sqrt{3}\ \text{ft}^2}$

An isosceles triangle has a $\boxed{\text{perimeter } 12\text{ m.}}$

Find all sides that gives max. Area




$P = 12$ $\quad \boxed{y + 2x = 12}$

$$\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}$$

Heron's Formula $\quad S = \frac{P}{2} = 6$

$$\text{Area} = \sqrt{6(6-y)(6-x)(6-x)}$$

If radicand is max., then area is max.

Need to maximize $6(6-y)(6-x)^2$

$y + 2x = 12$

$y = 12 - 2x$

Need to maximize

$$f(x) = 6(6-(12-2x))(6-x)^2$$

$$= 6(2x-6)(6-x)^2 = 12(x-3)(6-x)^2$$

Max. happens where $f'(x) = 0$ and $f''(x) < 0$

$$f(x) = 12(x-3)(6-x)^2$$

$$f'(x) = 12\left[1 \cdot (6-x)^2 + (x-3)\cdot 2(6-x)^1 \cdot (-1)\right]$$

$$f'(x) = 12(6-x)\left[6-x-2(x-3)\right]$$

$$= 12(6-x)(-3x+12) = -36(6-x)(x-4)$$

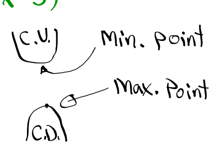
$f'(x) = 0 \rightarrow x = 6,\ x = 4$

$$f''(x) = -36\left[-1(x-4) + (6-x)\cdot 1\right] = -36\left[-x+4+6-x\right]$$

$$= -36(-2x+10) = 72(x-5)$$

$f''(6) = 72(6-5) = 72 > 0$

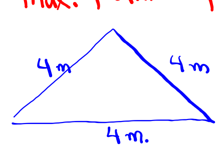
$f''(4) = 72(4-5) = -72 < 0$



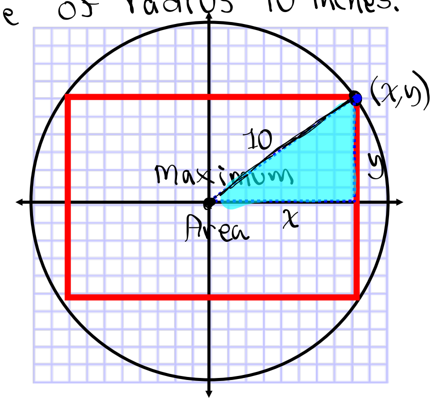

Max. point happens where $x = 4$

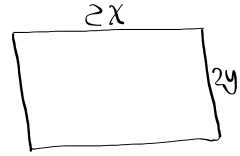
$y = 12 - 2x = 12 - 2(4)$

$\boxed{y = 4}$



Find dimensions of a rectangle with maximum area that can be inscribed in a Circle of radius 10 inches.




Maximize

Area $= 4xy$

$x^2+y^2=10^2$

$y^2 = 100 - x^2$

$y = \sqrt{100-x^2}$

$A(x) = 4x\sqrt{100-x^2}$

$A'(x) =$

$A''(x) =$

Make Sure to Find $A'(x)$ & $A''(x)$ by Wed.

$A(x) = 4x\sqrt{100-x^2}$

$A'(x) = 4\cdot\left[1\cdot\sqrt{100-x^2} + x\cdot\frac{-x}{\sqrt{100-x^2}}\right]$

$A'(x) = 4\cdot\frac{\sqrt{100-x^2}\cdot\sqrt{100-x^2} - x^2}{\sqrt{100-x^2}}$

$A'(x) = \frac{4\left[100-x^2-x^2\right]}{\sqrt{100-x^2}}$

$A'(x) = \frac{8(50-x^2)}{\sqrt{100-x^2}}$

$\frac{d}{dx}\left[\sqrt{100-x^2}\right] =$

$\frac{d}{dx}\left[(100-x^2)^{1/2}\right] =$

$\frac{1}{2}\cdot(100-x^2)^{-1/2}\cdot(-2x) =$

$\frac{-x}{\sqrt{100-x^2}}$

$A'(x) = 0$

$50 - x^2 = 0$ $\quad x^2 = 50$

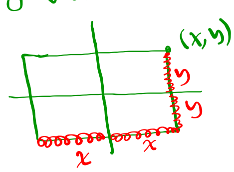
$x = \pm 5\sqrt{2}$

| $A'(x)>0$ | | $A'(x)<0$ |
|---|---|---|
| $x=7$ | $5\sqrt{2}$ $\approx 7.1$ | $x=8$ |

Max.

Max. happens when $x = 5\sqrt{2}$

$y = \sqrt{100-x^2} = 5\sqrt{2}$




Dimensions $2x$ by $2y$

$10\sqrt{2}$ in. by $10\sqrt{2}$ in.

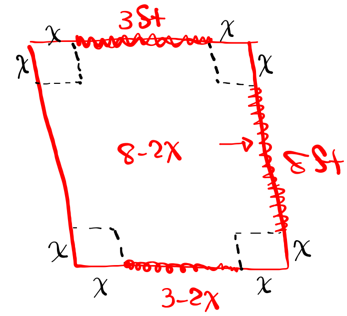
Max. Area ⇒ $4xy$

$= 4\cdot 5\sqrt{2}\cdot 5\sqrt{2} = 100\sqrt{4} = \boxed{200 \text{ in}^2}$

A sheet of metal has a rectangular shape, and it is 3 ft by 8 ft.

Cut 4 equal size squares from 4 corners.

Now bend up sides to make an open-top box.

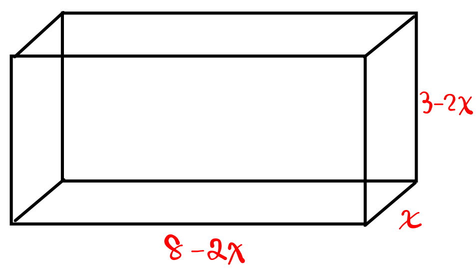



Volume of this Box:

$V = LWH$

$V = (8-2x)(3-2x)\cdot x$

Find $x$ that makes Max. Volume.




$V = (8-2x)(3-2x)\cdot x$

$= (24 - 16x - 6x + 4x^2)x$

$= (4x^2 - 22x + 24)x$

$V(x) = 4x^3 - 22x^2 + 24x$

$V'(x) = 12x^2 - 44x + 24$ $\qquad V'(x) = 0$

$V''(x) = 24x - 44$

Solve $V'(x) = 0$, determine for which one

Max where $V'(x) = 0$

$V''(x) < 0$

$V'(x) = 12x^2 - 44x + 24 = 4(3x^2 - 11x + 6)$

$V''(x) = 24x - 44$

$V'(x) = 0$

$3x^2 - 11x + 6 = 0$

$(3x - 2)(x - 3) = 0$

or

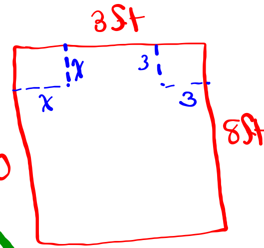
$3x - 2 = 0 \qquad x - 3 = 0$

$x = \frac{2}{3} \qquad \cancel{x = 3}$

$V''\left(\frac{2}{3}\right) =$

$\cancel{24}^{8}\left(\frac{2}{\cancel{3}}\right) - 44 =$

$16 - 44 = -\# < 0$




C.D.

Max. Point at $x = \frac{2}{3}$

Max. Volume $\underbrace{\frac{2}{3}\left(8 - 2\cdot\frac{2}{3}\right)\left(3 - 2\cdot\frac{2}{3}\right)}_{\to\ \#}$ $ft^3$

Volume

$V(x) = (8 - 2x)(3 - 2x)\cdot x$

Dimensions: $\frac{2}{3}, 8 - 2\cdot\frac{2}{3}, 3 - 2\cdot\frac{2}{3}$

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Find a number on the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$ Such that the Sum of the number and its reciprocal is as Small as possible.

Let $x$ be the number on $\left[\frac{1}{2}, \frac{3}{2}\right]$

Minimize $x + \frac{1}{x}$ $\Rightarrow$ Minimize $f(x) = x + \frac{1}{x}$

$f\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{1}{2} + 2 = 2.5$

$f\left(\frac{3}{2}\right) = \frac{3}{2} + \frac{1}{\frac{3}{2}} = 1.5 + \frac{2}{3} = 1.5 + .666\ldots = 2.1\overline{6}$

$f(x) = x + x^{-1}$

$f'(x) = 1 - x^{-2}$ $\to$ $1 - \frac{1}{x^2} = 0 \qquad x^2 - 1 = 0$

$x = \pm 1$

$f''(x) = 2x^{-3} = \frac{2}{x^3}$

$x = -1$ is not in $\left[\frac{1}{2}, \frac{3}{2}\right]$

what about $x = 1$?

$f''(1) = \frac{2}{1^3} = 2 > 0$

C.U.

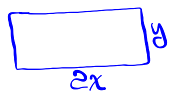
$f(1) = 1 + \frac{1}{1} = \boxed{2}$

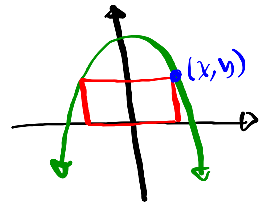
$x = 1$ Min. Point

the number is 1.

April 20, 2022

A rectangle is sitting on the x-axis and its two upper corners are on the curve given by $y = 16 - x^2$.




Find the dimensions with max. Area.

$A(x) = 2x(16 - x^2)$

$A(x) = 32x - 2x^3$

$A'(x) = 32 - 6x^2$

$A'(x) = 0$

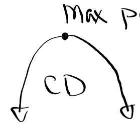
$x^2 = \frac{32}{6}$

$x^2 = \frac{16}{3}$

$x = \frac{4}{\sqrt{3}}$

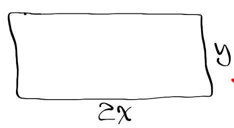
$x = \frac{4\sqrt{3}}{3}$

Max point




$A''(x) = -12x$

$A''\left(\frac{4\sqrt{3}}{3}\right) = -\# < 0$




$\boxed{\text{Dimensions are } \frac{8\sqrt{3}}{3} \text{ by } \frac{32}{3}}$

$y = 16 - x^2 = 16 - \frac{16}{3} = \frac{32}{3}$

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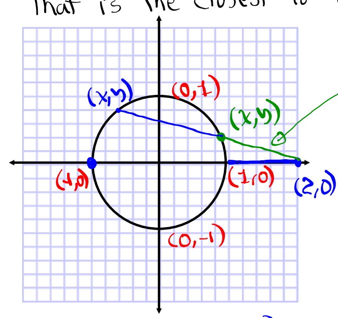
Find a point on the circle $x^2 + y^2 = 1$ that is the closest to $(2,0)$

Ans: $(1,0)$




$d = \sqrt{(x-2)^2 + (y-0)^2}$

Review distance formula

$d = \sqrt{(x-2)^2 + y^2}$

$= \sqrt{(x-2)^2 + 1 - x^2}$

$= \sqrt{x^2 - 4x + 4 + 1 - x^2}$

$= \sqrt{\underline{5 - 4x}}$

we need to minimize

$f(x)\ 5 - 4x$

$f'(x) = -4$

We are working on $[-1, 1]$

Decreasing

$f(-1) = 5 - 4(-1) = 5 + 4 = 9$

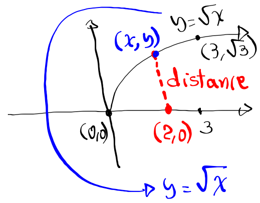
distance $= \sqrt{9} = \boxed{3}$

$f(1) = 5 - 4(1) = 5 - 4 = 1$

distance $= \sqrt{1} = \boxed{1}$ ← Min. distance at $x = 1$

closest point to $(2,0)$ is $(1,0)$.

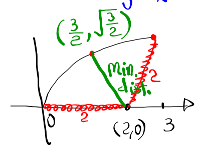
Find all points on $y=\sqrt{x}$ on $[0,3]$ that are closest to, and at the greatest distance from $(2,0)$.




$$d=\sqrt{(x-2)^2+(y-0)^2}$$
$$=\sqrt{(x-2)^2+y^2}$$
$$=\sqrt{(x-2)^2+x}$$

$y=\sqrt{x}$

$y^2=x$




$f(x)=(x-2)^2+x$

$f(0)=4 \rightarrow$ distance $=\sqrt{4}=2$ ✓

$f(3)=4 \rightarrow$ distance $=\sqrt{4}=2$

$f'(x)=2(x-2)^1\cdot 1+1 = 2x-4+1 = 2x-3$

$f''(x)=2>0$ C.U. CU Min at $2x-3=0$, $x=\frac{3}{2}$

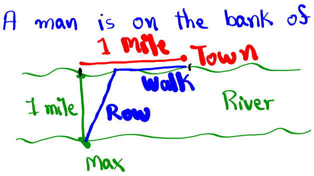
Max dist $(0,0),(3,\sqrt{3})$ $d=2$

Min dist $(\frac{3}{2},\sqrt{\frac{3}{2}})$ $d=\frac{\sqrt{7}}{2}$

$$d=\sqrt{(x-2)^2+x}$$

$$\sqrt{\left(\frac{3}{2}-2\right)^2+\frac{3}{2}}=\sqrt{\frac{1}{4}+\frac{3}{2}}=\sqrt{\frac{1}{4}+\frac{6}{4}}=\sqrt{\frac{7}{4}}=\frac{\sqrt{7}}{2}$$

A man is on the bank of river that is 1 mile wide



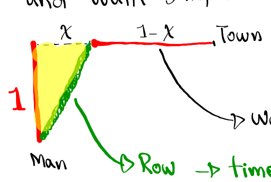

A town is 1 mile upstream on the opposite bank.

He intends to row on a straight line, then walk to the city.

To what point he should row to and walk to town in $\boxed{\text{least amount of time}}$ if he can row 4 mph and walk 5 mph?

Speed $=\frac{\text{distance}}{\text{time}}$

time $=\frac{\text{distance}}{\text{speed}}$




Walk $\rightarrow$ time $=\frac{1-x}{5}$

Row $\rightarrow$ time $=\frac{\sqrt{x^2+1}}{4}$

Total Time $\rightarrow \frac{\sqrt{x^2+1}}{4}+\frac{1-x}{5}=f(x)$

Minimize this

$$f(x) = \frac{1}{4}(x^2+1)^{1/2} + \frac{1-x}{5}$$

$$f'(x) = \frac{1}{4}\cdot\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x - \frac{1}{5}$$

$$= \frac{1}{\cancel{8}\,4}\cdot\frac{\cancel{2}x}{\sqrt{x^2+1}} - \frac{1}{5} = \frac{5x - 4\sqrt{x^2+1}}{20\sqrt{x^2+1}}$$

Solve $f'(x)=0$

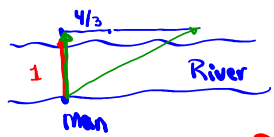
$$5x - 4\sqrt{x^2+1} = 0$$
$$5x = 4\sqrt{x^2+1}$$
$$25x^2 = 16(x^2+1)$$
$$25x^2 = 16x^2 + 16$$
$$9x^2 = 16$$
$$x^2 = \frac{16}{9} \quad \boxed{x = \frac{4}{3}}$$

Larger than 1.

$f'<0$ | $f'>0$

$x=0$ | $\frac{4}{3}$ | $x=2$

Min. Point




$x=0$ $\quad f(0) = \frac{1}{4} + \frac{1}{5} = \frac{9}{20}$

$x=1$ $\quad f(1) = \frac{\sqrt{2}}{4} + 0 = \frac{\sqrt{2}}{4}$

Row to the town

One week from Today $\Rightarrow$ Exam 2