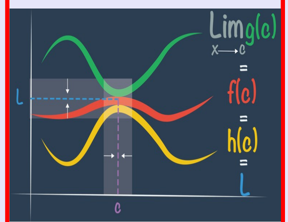
Math 261

Spring 2022

Lecture 16



Class QZ 10

Use linear approximate to estimate $\sqrt[3]{1.1}$. Round to 3-decimals.

$f(x) = \sqrt[3]{x}$ $\quad a = 1$ $\quad f(1) = 1$

$f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$ $\quad f'(1) = \dfrac{1}{3}$

$\boxed{L(x) = f(1) + f'(1)(x-1)}$

$L(x) = 1 + \frac{1}{3}(x-1)$

$L(1.1) = 1 + \frac{1}{3}(1.1-1) = 1 + \frac{1}{30}$

$= \boxed{\frac{31}{30}} \approx \boxed{1.033}$

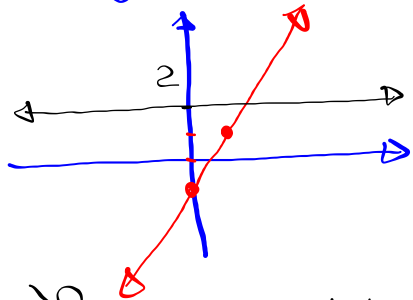
Consider $f(x) = 2x - 1$

$\frac{2}{1}$ $\frac{\text{Rise}}{\text{Run}}$

1) Linear function, $m = 2$, Y-Int $(0, -1)$

2) Since $m > 0$, It is increasing

3) $f'(x) = 2$



think of $f'(x)$ as slope of tangent line

Since $f'(x) > 0 \Rightarrow m_{\text{tan. line}} > 0 \Rightarrow f(x)$ must be increasing.

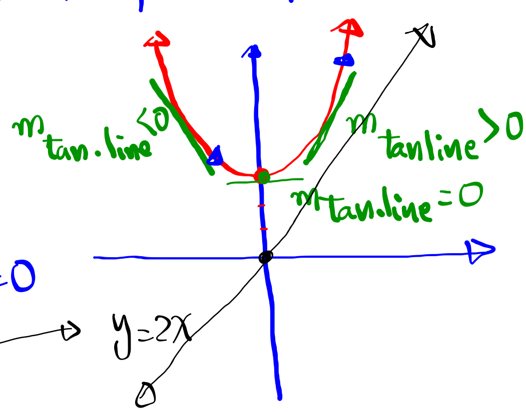
Consider $f(x) = x^2 + 3$

1) Parabola, Vertex $(0, 3)$, opens up

$f(x)$ is decreasing $(-\infty, 0)$

$f(x)$ is increasing $(0, \infty)$

$f(x)$ is stationary at $x = 0$



2) $f'(x) = 2x$

$2x = 0$ $\quad x = 0$

$f'(x) < 0$ $\quad f'(x) > 0$

$f(x)$ Decreasing $\quad 0 \quad f(x)$ increasing

$f(x) = x^3$

1) Domain $(-\infty, \infty)$

2) $f(x)$ is increasing on $(-\infty, \infty)$

3) $f'(x) = 3x^2 \geq 0$

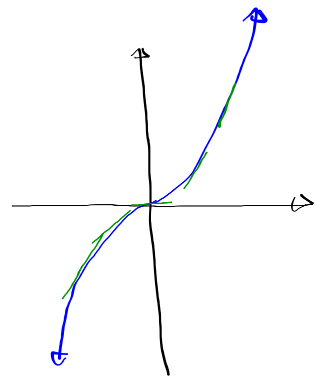
$3x^2 = 0 \Rightarrow x = 0$

$f'(x) > 0$ $\quad$ $f'(x) > 0$

0

$f(x)$ increasing $\quad$ $f'(x) = 0$ $\quad$ $f(x)$ increasing

Horizontal tan. line



$f'(x) > 0 \iff f(x)$ increasing

$f'(x) < 0 \iff f(x)$ decreasing

$f'(x) = 0 \iff f(x)$ has horizontal tan. line

Consider $f(x) = 2x^3 + 3x^2 - 36x$

1) factor $f(x)$ $\quad$ $f(x) = x(2x^2 + 3x - 36)$

$= \boxed{x(2x^2 + 3x - 36)}$

2) Find all intercepts

Y-Int $\rightarrow$ Let $x = 0$, find $f(0)$ $\Rightarrow f(0) = 0 \Rightarrow (0, 0)$

x-Int $\rightarrow$ Let $y = 0$, $f(x) = 0$, $x(\underbrace{2x^2 + 3x - 36}_{\text{use Q-formula}}) = 0$

$x = 0$ $\quad$ $x = ?$ $\quad$ $x = ?$

$(0, 0)$ $\quad$ $\boxed{(?, 0), (?, 0)}$

3) Find $f'(x)$, factor it completely.

$f'(x) = 6x^2 + 6x - 36$ $\quad$ $f'(x) = 6(x^2 + x - 6)$

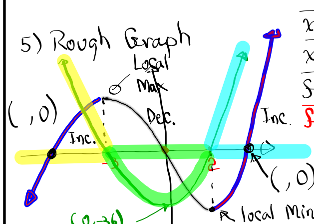
Parabola open upward $\quad$ $f'(x) = 6(x + 3)(x - 2)$

4) Let's Solve $f'(x) = 0$, $\Rightarrow x = -3$, $x = 2$

make Sign chart

x	$-\infty$ to -3	-3 to 2	2 to ∞
6	+	+	+
$x+3$	−	+	+
$x-2$	−	−	+
$f'(x)$	+	−	+
$f(x)$	Inc.	Dec.	Inc.

5) Rough Graph



Consider $f(x) = x^4 - 2x^2 + 3$

1) Polynomial function $\Rightarrow$ Domain $(-\infty, \infty)$

2) Y-Int $\rightarrow (0,3)$

3) x-Int $\rightarrow f(x)=0 \rightarrow x^4 - 2x^2 + 3 = 0$

No real solutions

No x-Ints.

4) $f'(x) = 4x^3 - 4x$ $\quad f'(x) = 4x(x^2-1)$

$f'(x) = 4x(x+1)(x-1)$

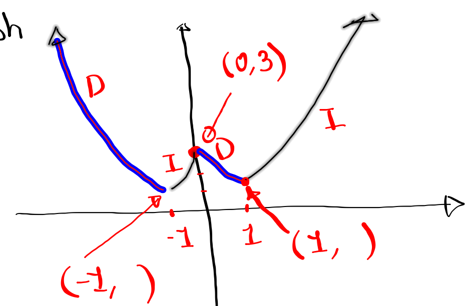
$f'(x) = 0 \Rightarrow x=0,\ x=-1,\ x=1$

5) Sign Chart

x	$-\infty$ … -1	-1 … 0	0 … 1	1 … ∞
4	+	+	+	+
x	−	−	+	+
$x+1$	−	+	+	+
$x-1$	−	−	−	+
$f'(x)$	−	+	−	+
$f(x)$	Dec.	Inc.	Dec.	Inc

Local Min (at -1), Local Max (at 0), Local Min (at 1)

6) Rough Graph



First Derivative:

helps us with increasing / Decreasing Intervals.

Second Derivative:

helps us with concavity of the graph.

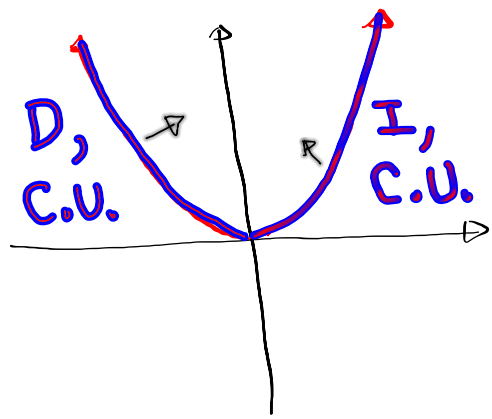
$f''(x) > 0 \iff f(x)$ is concave up. $\cup$

$f''(x) < 0 \iff f(x)$ " " down $\cap$

Consider $f(x) = x^2$

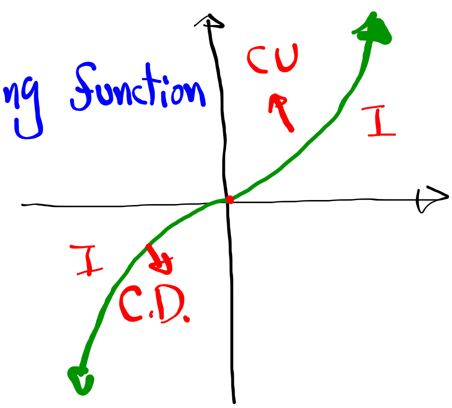
$f'(x) = 2x$

$f''(x) = 2 > 0 \Rightarrow f(x)$ is concave up.



$f(x) = x^3$

$f'(x) = 3x^2 > 0 \Rightarrow$ Increasing function



$f''(x) = 6x$

$6x = 0 \Rightarrow x = 0$

$-\infty$	$f''<0$ C.D.	0	$f''>0$ C.U.	∞

Consider $f(x) = x^4 - 4x^3$

1) Domain $(-\infty, \infty)$

2) Y-Int $(0,0)$

3) X-Int $(0,0), (4,0)$

$f(x) = 0$

$x^4 - 4x^3 = 0$

$x^3(x-4) = 0$

4) Find $f'(x)$, Factor.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

5) Find $f''(x)$, Factor.

$f''(x) = 12x^2 - 24x = 12x(x-2)$

6) Solve $f'(x) = 0$

$\boxed{x = 0 \text{ or } x = 3}$

7) Solve $f''(x) = 0$

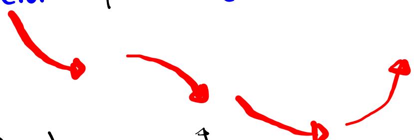
$\boxed{x = 0 \text{ or } x = 2}$

8) Sign Chart

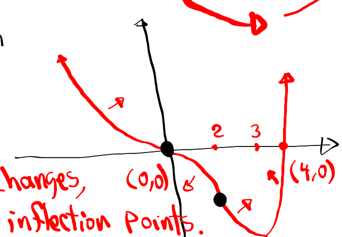
$f'(x) = 4x^2(x-3)$

$f''(x) = 12x(x-2)$

x	$-\infty$ to 0	0 to 2	2 to 3	3 to ∞
$f'(x)$	$-$	$-$	$-$	$+$
$f''(x)$	$+$	$-$	$+$	$+$
$f(x)$	Dec. C.U.	Dec. C.D.	Dec. C.U.	Inc. C.U.



9) Rough Graph



Not scaled

where concavity changes, we have inflection points.

$(0,0), (2,-16)$

$f(2) = 2^4 - 4(2)^3$

Suppose $f'(0) = f'(2) = f'(4) = 0$

$f'(x) > 0$ if $x < 0$ or $2 < x < 4$

$f'(x) < 0$ if $0 < x < 2$ or $x > 4$

$f''(x) > 0$ if $1 < x < 3$

$f''(x) < 0$ if $x < 1$ or $x > 3$

x	$-\infty$		0		1		2		3		4		∞
$f'(x)$		$+$		$-$		$-$		$+$		$+$		$-$	
$f''(x)$		$-$		$-$		$+$		$+$		$-$		$-$	
$f(x)$		↗		↘		↘		↗		↗		↘	

Local max $x = 0$, $x = 4$

Local Min $x = 2$

Inflection Points at $x = 1$, $x = 3$

Given $f(x) = x^3 - 12x + 2$

1) Find $f'(x)$, and Solve $f'(x) = 0$ → $x = 2$, $x = -2$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$

2) Find $f''(x)$, and Solve $f''(x) = 0$

$f''(x) = 6x$ $\quad x = 0$

3) Set-up the Sign chart, discuss increasing/Decreasing, and Concavity.

x	$-\infty$		-2		0		2		∞
$f'(x)$		$+$		$-$		$-$		$+$	
$f''(x)$		$-$		$-$		$+$		$+$	
$f(x)$		↗		↘		↘		↗	

Inc: $(-\infty, -2), (2, \infty)$

Dec: $(-2, 2)$

Con. Up $(0, \infty)$ Con. Down $(-\infty, 0)$

Local max at $x = -2$, Local Min at $x = 2$.

Inflection Point at $x = 0$.

Consider $f(x) = x - \sin x$ on $[0, 2\pi]$

1) Find $f'(x)$, Solve $f'(x) = 0$ on the given interval.

$f'(x) = 1 - \cos x$ $\quad$ $1 - \cos x = 0$ $\quad$ $\cos x = 1$

$x = 0, \; x = 2\pi$

2) Find $f''(x)$, Solve $f''(x) = 0$ on the given interval.

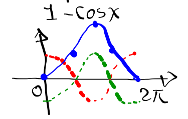
$f''(x) = 0 - (-\sin x) = \sin x$ $\quad$ $\sin x = 0$

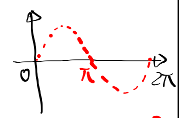
$x = 0 \quad x = 2\pi$

3) Set-up the Sign chart, discuss inc./Dec./Concavity as well as local max, Min, and inflection points.

x	0		π		2π
$f'(x)$	+	+		+	+
$f''(x)$	+	+		−	−
$f(x)$					

$1 - \cos x$





No local Max or Min.

C.U. $(0, \pi)$, C.D. $(\pi, 2\pi)$

$f(\pi) = \pi - \cancel{\sin \pi}^{\,0}$
$= \pi$

Inflection Point $(\pi, f(\pi)) = (\pi, \pi)$

If $f(x)$ and $g(x)$ are both Concave Up Show that $(f + g)(x)$ is also Concave Up.

Since $f(x)$ and $g(x)$ are C.U. $\Rightarrow f''(x) > 0$,
$g''(x) > 0$

So $f''(x) + g''(x) > 0 \Rightarrow f(x) + g(x)$ are C.U.

Suppose $f(x)$ and $g(x)$ are possitive, increasing, concave up functions on interval I.

$f(x)>0$, $g(x)>0$, $f'(x)>0$, $g'(x)>0$, $f''(x)>0$, $g''(x)>0$

Show that fg is also concave up on I.

need $\frac{d^2}{dx^2}[fg]$

Let $h(x)=f(x)\cdot g(x)$

$h'(x)=\overset{+}{f'(x)}\,\overset{+}{g(x)}+\overset{+}{f(x)}\,\overset{+}{g'(x)}>0$ $h(x)$ is increasing.

$h''(x)=f''(x)\,g(x)+f'(x)\,g'(x)+f'(x)\,g'(x)+f(x)\cdot g''(x)$

$=\underset{+}{f''}\,\underset{+}{g}+\underset{+++}{2f'g'}+\underset{+}{f}\,\underset{+}{g''}>0$

$h''(x)>0 \Rightarrow h(x)$ is concave up

$\boxed{fg \text{ " " "}}$