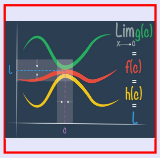
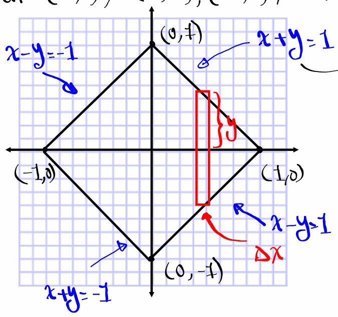
Math 261

Fall 2023

Lecture 56

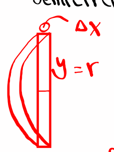


The base of a solid is a square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$.



Cross-Sections $\perp$ x-axis

Cross Sections are Semicircle.



Volume $= \frac{\pi r^2}{2} \cdot \Delta x$

$$V = \int_{-1}^{1} \frac{\pi r^2}{2}\,dx = \frac{\pi}{\cancel{2}} \cdot \cancel{2}\int_0^1 (1-x)^2\,dx$$

$$= \pi \int_0^1 (1-x)^2\,dx$$

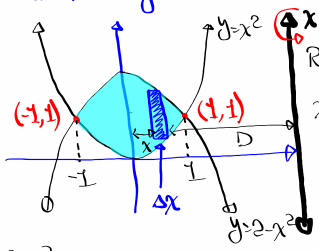
$u = 1-x$ $\quad$ $x=0 \to u=1$

$du = -dx$ $\quad$ $x=1 \to u=0$

$-du = dx$

$$= \pi \int_1^0 u^2 \cdot -du = \pi \int_0^1 u^2\,du = \pi \cdot \frac{u^3}{3}\Big|_0^1 = \boxed{\frac{\pi}{3}}$$

Consider the region bounded by $y=x^2$ and $y=2-x^2$



Rotate this region by $x=3$, Set-up the integral for the volume.

My Ref. Rect. is Parallel to A.O.R.

Shell Method

$x+D=3$

$D=3-x$

$2\pi D H \Delta x$

Distance from A.O.R.

Height of Ref. Rect.

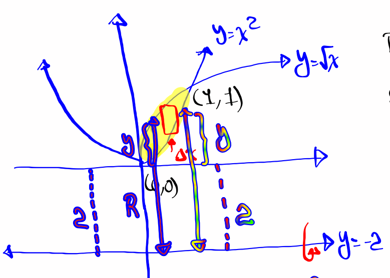
Top − Bottom

$2-x^2-x^2$

$=2-2x^2$

$$V=\int_{-1}^{1} 2\pi(3-x)(2-2x^2)\,dx$$

Consider the bounded region by $y=\sqrt{x}$ and $y=x^2$.



Rotate about $y=-2$.

Set-up the integral for the volume.

Ref. Rect. ⊥ A.O.R.

Ref. Rect. is not totally attached to A.O.R.

Washer Method

$\pi[R^2-r^2]\Delta x$

$R = y+2 = \sqrt{x}+2$ $\quad r = x^2+2$

$$V=\int_{0}^{1} \pi\left[(\sqrt{x}+2)^2-(x^2+2)^2\right]dx$$

Find f_{ave} for $f(x) = x\sec^2 x^2$ from $x=0$ to $x=\sqrt{\frac{\pi}{4}}$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\sqrt{\frac{\pi}{4}}-0}\int_0^{\sqrt{\pi/4}} x\sec^2 x^2\,dx$$

$$= \frac{1}{\frac{\sqrt{\pi}}{2}}\int_0^{\sqrt{\frac{\pi}{4}}} x\sec^2 x^2\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$x=0 \quad u=0$

$x=\sqrt{\frac{\pi}{4}} \quad u=\frac{\pi}{4}$

$$= \frac{\cancel{2}}{\sqrt{\pi}}\int_0^{\pi/4} \sec^2 u\,\frac{du}{\cancel{2}}$$

$$= \frac{1}{\sqrt{\pi}}\cdot \tan u\Big|_0^{\frac{\pi}{4}}$$

$$= \boxed{\frac{1}{\sqrt{\pi}}}$$

Suppose $f'(x)$ is cont. on $[a,b]$, show

$$2\int_a^b f(x)\,f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

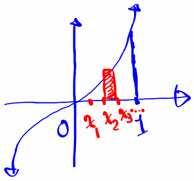
$u = f(x) \qquad x=a \to u=f(a)$

$du = f'(x)\,dx \qquad x=b \to u=f(b)$

$$2\int_{f(a)}^{f(b)} u\,du = \cancel{2}\cdot\frac{u^2}{\cancel{2}}\Big|_{f(a)}^{f(b)} = u^2\Big|_{f(a)}^{f(b)}$$

$$= [f(b)]^2 - [f(a)]^2$$

Evaluate $\lim_{n\to\infty} \frac{1}{n}\left[\underbrace{\left(\frac{1}{n}\right)^9+\left(\frac{2}{n}\right)^9+\left(\frac{3}{n}\right)^9+\cdots+\left(\frac{n}{n}\right)^9}\right]$



$f(x) = x^9$

$[0, 1]$

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$

$x_i = a + i\Delta x$

$= 0 + \frac{i}{n}$

$R_i = f(x_i)\cdot \Delta x$

$= f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \left(\frac{i}{n}\right)^9\cdot\frac{1}{n}$

$$\lim_{n\to\infty}\sum_{i=1}^{n} R_i = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^9\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^9$$

$$= \int_0^1 f(x)\,dx = \int_0^1 x^9\,dx$$

$$= \frac{x^{10}}{10}\Big|_0^1 = \boxed{\frac{1}{10}}$$

Evaluate $\int \frac{dx}{\cos^2 x\sqrt{1+\tan x}}$

$u = 1 + \tan x$

$du = \sec^2 x\,dx$

$du = \frac{1}{\cos^2 x}\,dx$

$$= \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du$$

$$= \frac{u^{1/2}}{1/2} + C = 2\sqrt{u} + C$$

$$= \boxed{2\sqrt{1+\tan x} + C}$$

Evaluate $\int_0^R x\sqrt{R^2-x^2}\,dx$

Hint: Let $u=R^2-x^2$

$du=-2x\,dx$

$\frac{du}{-2}=x\,dx$

$x=0 \quad u=R^2$

$x=R \quad u=0$

$= \int_{R^2}^{0} \sqrt{u}\ \frac{du}{-2}$

$= \frac{1}{2}\int_0^{R^2} u^{1/2}\,du$

$= \frac{1}{\cancel{2}} \cdot \frac{u^{3/2}}{\frac{3}{\cancel{2}}}\Big|_0^{R^2} = \frac{1}{3}\,u\sqrt{u}\,\Big|_0^{R^2}$

$= \frac{1}{3}R^2\sqrt{R^2} - 0 = \boxed{\frac{R^3}{3}}$

Suppose $f(x)$ is continuous and $\int_0^9 f(x)\,dx=4$

Find $\int_0^3 x\, f(x^2)\,dx$

Hint: Let $u=x^2$

$du=2x\,dx$

$\frac{du}{2}=x\,dx$

$x=0 \quad u=0$

$x=3 \quad u=9$

$\int_0^9 f(u)\,\frac{du}{2} = \frac{1}{2}\int_0^9 f(u)\,du$

$= \frac{1}{2}\cdot 4 = \boxed{2}$

$$\int_0^1 \frac{dx}{(\underbrace{1+\sqrt{x}}_{u})^4}$$

$u = 1+\sqrt{x}$ $\quad x=0 \quad u=1$

$u - 1 = \sqrt{x}$ $\quad x=1 \quad u=2$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$2(u-1)\,du = dx$

$$\int_1^2 \frac{2(u-1)}{u^4}\,du$$

$$= 2\int_1^2 \frac{u-1}{u^4}\,du$$

$$= 2\int_1^2 \left(\frac{u}{u^4} - \frac{1}{u^4}\right)du = 2\int_1^2 (u^{-3} - u^{-4})\,du$$

$$= 2\left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3}\right]\Bigg|_1^2 = 2\left[\frac{1}{3u^3} - \frac{1}{2u^2}\right]\Bigg|_1^2$$

$$= 2\underbrace{\left[\left(\frac{1}{24} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]}_{1/12} = 2\cdot\frac{1}{12} = \boxed{\frac{1}{6}}$$