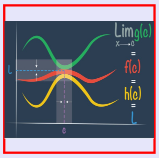
Math 261

Fall 2023

Lecture 51



Feb 19-8:47 AM

Mean - Value Theorem

1) $f(x)$ is cont. on $[a,b]$

2) $f(x)$ is diff. on (a,b)

then there is a number c in (a,b)

such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

MVT for integration

If $f(x)$ is cont. on $[a,b]$, there is at least a number c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$f(x) = x^2$ on $[1,4]$

$$\int_1^4 f(x)\,dx = f(c)(4-1)$$

$$\int_1^4 x^2\,dx = c^2 \cdot 3$$

$$\left.\frac{x^3}{3}\right|_1^4 = 3c^2$$

$$\frac{1}{3}(4^3 - 1^3) = 3c^2$$

$$\frac{63}{3} = 3c^2$$

$c^2 = 7 \rightarrow c = \sqrt{7}$ $\in [1,4]$

Nov 30-10:26 AM

$f(x) = x^2 - 2x \;,\; [1,3]$

Cont. everywhere

$\int_1^3 (x^2-2x)\,dx = (c^2-2c)(3-1)$

$\left(\frac{x^3}{3} - x^2\right)\Big|_1^3 = 2(c^2-2c)$

$\left(\frac{3^3}{3} - 3^2\right) - \left(\frac{1^3}{3} - 1^2\right) = 2(c^2-2c)$, with $\frac{3^3}{3} - 3^2 \to 0$

$\frac{2}{3} = 2(c^2-2c)$

$c^2 - 2c + 1 = \frac{1}{3} + 1$

$(c-1)^2 = \frac{4}{3}$

$c - 1 = \pm\sqrt{\frac{4}{3}}$

$c = 1 \pm \frac{2}{\sqrt{3}}$

$c = 1 \pm 1.2$

$\boxed{c = 2.2}$ or $c = -.2$

Nov 30-10:33 AM

If $f(x)$ is integrable on $[a,b]$, then the average value (Mean Value) of $f(x)$ on $[a,b]$ is

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

$f(x) = \sqrt{x} \;,\; [4,9]$

$f_{ave} = \frac{1}{9-4}\int_4^9 \sqrt{x}\,dx = \frac{1}{5}\int_4^9 x^{1/2}\,dx$

$= \frac{1}{5}\cdot\frac{x^{3/2}}{3/2}\Big|_4^9 = \frac{2}{15}\cdot x\sqrt{x}\Big|_4^9$

$= \frac{2}{15}\left[9\sqrt{9} - 4\sqrt{4}\right]$

$= \frac{2}{15}\left[27 - 8\right] = \boxed{\frac{38}{15}}$

Nov 30-10:38 AM

Find f_{ave} for $\underbrace{f(x) = \sin x}$ over $[\underset{a}{0}, \underset{b}{2\pi}]$

Cont. everywhere

$\therefore$ diff. everywhere

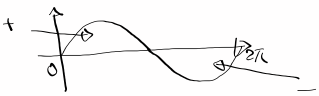
$\therefore$ integrable everywhere

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2\pi - 0}\int_0^{2\pi} \sin x\,dx$$

$$= \frac{1}{2\pi}\cdot\left(-\cos x\right)\Big|_0^{2\pi} = \frac{-1}{2\pi}\left[\overset{1}{\cos 2\pi} - \overset{1}{\cos 0}\right]$$

$$= \frac{-1}{2\pi}(0) = \boxed{0}\ \checkmark$$



Nov 30-10:43 AM

Find f_{ave} for $\underbrace{f(x) = \sec^2 x}$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Integrable on $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

$$f_{ave} = \frac{1}{\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)}\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = \frac{1}{\frac{\pi}{2}}\cdot \tan x\Big|_{-\frac{\pi}{4}}^{\pi/4}$$

$$= \frac{2}{\pi}\left[\tan\frac{\pi}{4} - \tan\left(\frac{-\pi}{4}\right)\right]$$

$$= \frac{2}{\pi}\left[1 - (-1)\right] = \boxed{\frac{4}{\pi}}$$

Nov 30-10:49 AM

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x)$$

$$\frac{d}{dx}\left[\int_{x}^{x^2} \frac{1}{t}\,dt\right] = \frac{1}{x^2}\cdot 2x - \frac{1}{x}\cdot 1$$

$$= \frac{2}{x} - \frac{1}{x} = \boxed{\frac{1}{x}}$$

$$\frac{d}{dx}\left[\int_{4}^{\sin x} \frac{1}{1+t^2}\,dt\right] = \frac{1}{1+\sin^2 x}\cdot \cos x - \frac{1}{1+4^2}\cdot 0$$

$$= \boxed{\frac{\cos x}{1+\sin^2 x}}$$

Nov 30-10:53 AM

$$F(x) = \int_2^x \sqrt{3t^2+1}\,dt$$

Evaluate $F(2) = \int_2^2 \sqrt{3t^2+1}\,dt = \boxed{0}$

Evaluate $F'(2)$

$$F'(x) = \sqrt{3x^2+1}\cdot 1 - \sqrt{3\cdot 2^2+1}\cdot 0$$

$$F'(x) = \sqrt{3x^2+1}$$

$$F'(2) = \sqrt{3\cdot 2^2+1} = \sqrt{13}$$

Nov 30-10:59 AM

Prove $F(x) = \int_{\boxed{x}}^{\boxed{3x}} \frac{1}{t}\,dt$ is constant on $(0,\infty)$.

$\frac{d}{dx}[\text{Constant}] = 0$

If we show $F'(x) = 0$, then F(x) must be constant

$F'(x) = \frac{1}{\cancel{3}x} \cdot \cancel{3} - \frac{1}{x} \cdot 1$

$= \frac{1}{x} - \frac{1}{x} = 0$

Nov 30-11:03 AM

Evaluate $\int_0^1 \sqrt[5]{1-2x}\,dx$

Hint: $u = 1-2x$

$du = -2\,dx$

$\frac{du}{-2} = dx$

$x=0 \to u=1$

$x=1 \to u=-1$

$\int_1^{-1} \sqrt[5]{u}\,\frac{du}{-2}$

$\frac{1}{5}+1 = \frac{6}{5}$

$= \frac{-1}{2}\int_1^{-1} u^{1/5}\,du = \frac{-1}{2} \cdot \frac{u^{\frac{6}{5}}}{\frac{6}{5}}\Big|_1^{-1}$

$= \frac{-5}{12}\, u\sqrt[5]{u}\Big|_1^{-1}$

$= \frac{-5}{12}\left[-1\sqrt[5]{-1} - 1\sqrt[5]{1}\right] = \frac{-5}{12}[1-1] = \boxed{0}$

$\boxed{\int_a^a f(x)\,dx = 0}$

$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$

Nov 30-11:08 AM

Evaluate

$$\int_0^{-3} \frac{x}{\sqrt{x^2+16}}\,dx$$

Hint let $u = x^2+16$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$x=0 \to u=16$

$x=-3 \to u=25$

$$\int_{16}^{25} \frac{1}{\sqrt{u}}\,\frac{du}{2}$$

$$= \frac{1}{2}\int_{16}^{25} u^{-1/2}\,du$$

$$= \cancel{\frac{1}{2}} \cdot \frac{u^{1/2}}{\cancel{1/2}}\Big|_{16}^{25} = \sqrt{u}\Big|_{16}^{25} = \sqrt{25}-\sqrt{16} = 5-4$$

$$= \boxed{1}$$

Nov 30-11:16 AM

Find $\int \frac{\cot^2 x}{\sin^2 x}\,dx$

$\cot x = \frac{\cos x}{\sin x}$

$$\frac{\cot^2 x}{\sin^2 x} = \frac{\frac{\cos^2 x}{\sin^2 x}}{\sin^2 x} = \frac{\cos^2 x}{\sin^4 x}$$

$$= \int \frac{\cos^2 x}{\sin^4 x}\,dx$$

$$= \int \frac{\cos^2 x}{\sin^2 x}\cdot\frac{1}{\sin^2 x}\,dx$$

$$= \int \cot^2 x \cdot \csc^2 x\,dx$$

$\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[\tan x] = \sec^2 x$

Let $u = \cot x$

$du = -\csc^2 x\,dx$

$-du = \csc^2 x\,dx$

$$= \int u^2 \cdot -du = -\int u^2\,du = -\frac{u^3}{3} + C$$

$$= \boxed{-\frac{1}{3}\cot^3 x + C}$$

Nov 30-11:22 AM