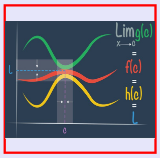
Math 261

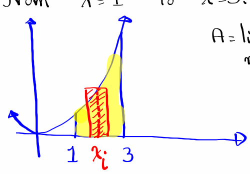
Fall 2023

Lecture 49



Feb 19-8:47 AM

Find the area below $f(x) = x^2$, above x-axis, from $x = 1$ To $x = 3$.



$$A = \lim_{n\to\infty} \boxed{\sum_{i=1}^{n} f(x_i)\cdot\Delta x}$$

Riemann Sum

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$x_i = a + i\cdot\Delta x = 1 + i\cdot\frac{2}{n} = 1 + \frac{2i}{n}$$

$$f(x_i) = \left(1 + \frac{2i}{n}\right)^2 = 1 + \frac{4i}{n} + \frac{4i^2}{n^2}$$

$$f(x_i)\cdot\Delta x = \left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right]\cdot\frac{2}{n} = \frac{2}{n} + \frac{8i}{n^2} + \frac{8i^2}{n^3}$$

$$\sum_{i=1}^{n} f(x_i)\cdot\Delta x = \sum_{i=1}^{n}\left[\frac{2}{n} + \frac{8i}{n^2} + \frac{8i^2}{n^3}\right]$$

$$= \sum_{i=1}^{n}\frac{2}{n} + \sum_{i=1}^{n}\frac{8i}{n^2} + \sum_{i=1}^{n}\frac{8i^2}{n^3}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{2}{n}\cdot n + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= 2 + \frac{8n^2 + \cdots}{2n^2} + \frac{16n^3 + \cdots}{6n^3}$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Nov 28-10:25 AM

$$= 2 + \frac{8n^2+\cdots}{2n^2} + \frac{16n^3+\cdots}{6n^3}$$

$$\lim_{n\to\infty}\left[2+\frac{8n^2+\cdots}{2n^2}+\frac{16n^3+\cdots}{6n^3}\right]$$

$$= 2 + \frac{8}{2} + \frac{16}{6} = 2+4+\frac{8}{3} = 6+\frac{8}{3}$$

$$= 6+2\tfrac{2}{3} = 8\tfrac{2}{3}$$

$$= \boxed{\frac{26}{3}}$$

If $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$ exist,

then $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x = \int_a^b f(x)\,dx$

In our example

$$\int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{1}{3}\left[3^3-1^3\right] = \frac{1}{3}\left[27-1\right] = \boxed{\frac{26}{3}}$$

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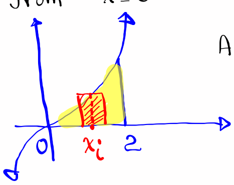
Find the area below $f(x)=x^3$, above x-axis from $x=0$ To $x=2$ using Riemann Sum.



$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\cdot\Delta x = 0 + i\cdot\frac{2}{n} = \frac{2i}{n}$$

$$f(x_i) = f\left(\frac{2i}{n}\right) = \left(\frac{2i}{n}\right)^3 = \frac{8i^3}{n^3}$$

$$f(x_i)\cdot\Delta x = \frac{8i^3}{n^3}\cdot\frac{2}{n} = \frac{16i^3}{n^4}$$

$$\sum_{i=1}^{n} f(x_i)\cdot\Delta x = \sum_{i=1}^{n}\frac{16i^3}{n^4} = \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{16}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2$$

$$= \frac{16n^4+\cdots}{4n^4}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$$= \lim_{n\to\infty}\left[\frac{16n^4+\cdots}{4n^4}\right]$$

$$= \frac{16}{4} = \boxed{4}$$

$$\int_0^2 x^3\,dx = \frac{x^4}{4}\Big|_0^2 = \frac{1}{4}\left[2^4-0^4\right] = \frac{16}{4} = \boxed{4}$$

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Evaluate

$$\int_1^2 (4x-2)^3\,dx$$

Use Subs. method

Let $u = 4x-2$ $\quad x=1 \rightarrow u=2$

$du = 4dx \quad x=2 \rightarrow u=6$

$\frac{du}{4} = dx$

$$\int_1^2 (4x-2)^3\,dx = \int_2^6 u^3 \cdot \frac{du}{4}$$

$$= \frac{1}{4}\cdot \frac{u^4}{4}\Big|_2^6 = \frac{1}{16}\left[6^4 - 2^4\right]$$

$$= \frac{1}{16}\cdot 1280 = \boxed{80}$$

Nov 28-10:57 AM

$$\int_{-5}^{0} x\sqrt{4-x}\,dx$$

$u = 4-x \longrightarrow x = 4-u$

$du = -dx$

$-du = dx$

$x=-5 \rightarrow u=9$

$x=0 \rightarrow u=4$

$$\int_{-5}^{0} x\sqrt{4-x}\,dx = \int_9^4 (4-u)\cdot\sqrt{u}\cdot -du$$

$$= -\int_9^4 (4\sqrt{u} - u\sqrt{u})\,du$$

$$= -\int_9^4 (4\cdot u^{1/2} - u^{3/2})\,du = -\left[4\cdot\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]\Big|_9^4$$

$$= -\left[\frac{8}{3}u\sqrt{u} - \frac{2}{5}u^2\sqrt{u}\right]\Big|_9^4$$

$$= -\left[\left(\frac{8}{3}\cdot 4\sqrt{4} - \frac{2}{5}\cdot 4^2\sqrt{4}\right) - \left(\frac{8}{3}\cdot 9\sqrt{9} - \frac{2}{5}\cdot 9^2\sqrt{9}\right)\right]$$

$$= -\left[\left(\frac{64}{3} - \frac{64}{5}\right) - \left(\frac{216}{3} - \frac{486}{5}\right)\right]$$

$$= -\left[\frac{-152}{3} + \frac{422}{5}\right] = \frac{152}{3} - \frac{422}{5} = \boxed{\frac{-506}{15}}$$

Nov 28-11:02 AM

$$\int_{-5}^{0} x\sqrt{4-x}\,dx$$

Let $u=\sqrt{4-x}$

$u^2=4-x \rightarrow x=4-u^2$

$2u\,du=-dx$

$-2u\,du=dx$

$x=-5 \rightarrow u=3$

$x=0 \rightarrow u=2$

$$\int_3^2 (4-u^2)\cdot u\cdot -2u\,du=\int_3^2 (u^2-4)\cdot 2u^2\,du$$

$$=2\int_3^2 (u^4-4u^2)\,du=2\left[\frac{u^5}{5}-\frac{4u^3}{3}\right]\Big|_3^2$$

$$=2\left[\left(\frac{2^5}{5}-\frac{4\cdot 2^3}{3}\right)-\left(\frac{3^5}{5}-\frac{4\cdot 3^3}{3}\right)\right]$$

$$=2\left[\left(\frac{32}{5}-\frac{32}{3}\right)-\left(\frac{243}{5}-\frac{108}{3}\right)\right]$$

$$=2\left[\frac{-211}{5}+\frac{76}{3}\right]=\boxed{\frac{-506}{15}}$$

Nov 28-11:13 AM

Evaluate $\displaystyle\int_0^{\pi/4}\sqrt{\tan x}\cdot\sec^2 x\,dx$ Hint: Let $u=\tan x$

$u=\tan x$

$du=\sec^2 x\,dx$

$x=0 \rightarrow u=0$

$x=\pi/4 \rightarrow u=1$

$$=\int_0^1 \sqrt{u}\,du=\int_0^1 u^{1/2}\,du$$

$$=\frac{u^{3/2}}{3/2}\Big|_0^1$$

$$=\frac{2}{3}u\sqrt{u}\Big|_0^1=\boxed{\frac{2}{3}}$$

what if we let $u=\sqrt{\tan x}$

$u^2=\tan x$

$2u\,du=\sec^2 x\,dx$

$$\int_0^1 u\cdot 2u\,du$$

$$=\int_0^1 2u^2\,du=2\cdot\frac{u^3}{3}\Big|_0^1=\boxed{2/3}$$

Nov 28-11:22 AM

Evaluate $\int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\, dx$

$2du = \frac{1}{\sqrt{x}}\, dx$

$x = \pi^2 \rightarrow u = \pi$

$x = 4\pi^2 \rightarrow u = 2\pi$

$$\int_{\pi}^{2\pi} \sin u \cdot 2du$$

$$= 2 \cdot -\cos u \Big|_{\pi}^{2\pi}$$

$$= -2\left[\cos 2\pi - \cos \pi\right]$$

$$= -2\left[1 - -1\right] = \boxed{-4}$$

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