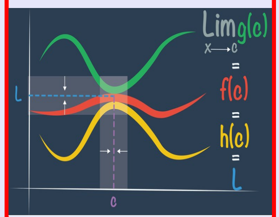
Math 261

Fall 2023

Lecture 15



Feb 19-8:47 AM

Class QZ 8

For $\varepsilon=.1$, find a $\delta>0$ such that $\lim_{x\to 2}(x^2+4x)=12$. ✓

$f(x)=x^2+4x$ $\qquad$ $\lim_{x\to 2}(x^2+4x)=2^2+4(2)=4+8=12$ ✓

$L=12$ ✓

$a=2$

For any $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|x^2+4x-12|<\varepsilon$ $\quad$ " $\quad$ $|x-2|<\delta$

$|(x+6)(x-2)|<\varepsilon$ $\quad$ " $\quad$ $|x-2|<\delta$

$|x+6|\,|x-2|<\varepsilon$

If $|x+6|<C$, then $|x-2|<\frac{\varepsilon}{C}$

If we wish $\delta\le 1$, then

$|x-2|<1$

$-1<x-2<1$

Add 8

$7<x+6<9$

$|x+6|<9 \leftarrow C$

Pick

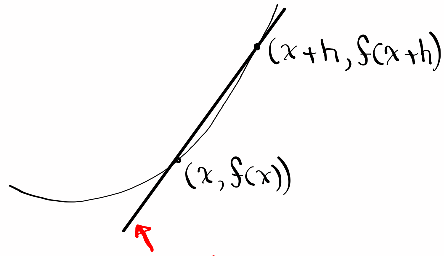
$\delta=\min\left\{1,\frac{\varepsilon}{9}\right\}$

For $\varepsilon=.1$

$\delta=\min\left\{1,\frac{.1}{9}\right\}$

$=\boxed{\frac{1}{90}}$

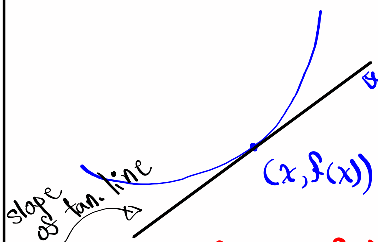
Sep 20-11:12 AM



Secant line

$$m = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

If $h \to 0$, then Secant line become tangent line



$$m = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

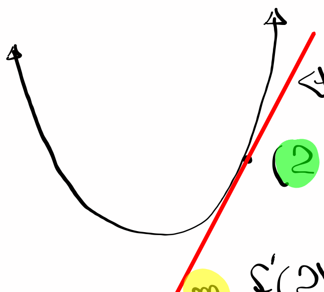
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

First derivative of $f(x)$

Sep 25-10:28 AM

Given $f(x) = x^2$

Find equation of the tangent line at $x=2$.



Point-Slope formula

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 4(x - 2)$$

$$(2, f(2)) = (2, 4)$$

Tangent Point

$$\boxed{y = 4x - 4}$$

$$m = f'(2) = 2(2) = 4$$

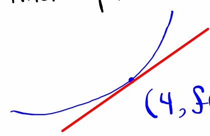
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \boxed{2x}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h)$$

Sep 25-10:33 AM

Given $f(x) = \dfrac{1}{\sqrt{x}}$ $\qquad f(4) = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$

Find eqn of the tan. line at $x=4$.



$m = f'(4) = \dfrac{-1}{2\cdot 4\cdot\sqrt{4}} = \boxed{\dfrac{-1}{16}}$

$(4, f(4)) = (4, \tfrac{1}{2})$

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = \frac{-1}{16}(x-4)$

$\vdots$

$\boxed{y = \ ?}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

LCD $= \sqrt{x+h}\,\sqrt{x}$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} = \frac{0}{0} \quad \text{I.F.}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{\cancel{x} - (\cancel{x} + h)}{\cancel{h}\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \quad -\cancel{h}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})}$$

$$= \frac{-1}{x\cdot 2\sqrt{x}} = \frac{-1}{2x\sqrt{x}} \quad f'(x)$$

Sep 25-10:40 AM

Given $f(x) = \dfrac{1}{x}$ $\qquad f(1) = \dfrac{1}{1} = 1$

Find equation of a normal line at $x = 1$.

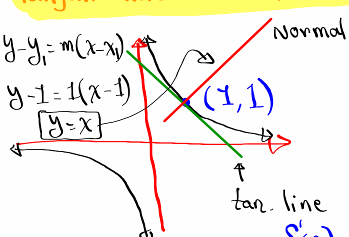
Normal line is a line perpendicular to the tangent line at tan. Point.

$m = \dfrac{-1}{f'(a)}$

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x-1)$

$\boxed{y = x}$



when two lines are $\perp$ to each other, Product of slopes is -1 except vertical & horizontal lines.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \qquad \text{LCD} = (x+h)\cdot x$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{h(x+h)\cdot x} = \lim_{h\to 0} \frac{-1}{(x+h)\cdot x} = \frac{-1}{x^2}$$

$f'(1) = \dfrac{-1}{1^2} = -1 \leftarrow$ Slope of tan. line

$\dfrac{-1}{f'(1)} = \dfrac{-1}{-1} = 1 \leftarrow$ Slope of normal line

Sep 25-10:51 AM

Let's look at some limits with trig. expressions

$$\boxed{\lim_{h\to 0} \frac{\sin h}{h} = 1} \qquad \& \qquad \boxed{\lim_{h\to 0} \frac{1-\cos h}{h} = 0}$$

Evaluate $\displaystyle\lim_{x\to 0} \frac{8\sin 8x}{8x} = 8 \lim_{x\to 0} \frac{\sin 8x}{8x} = 8\cdot 1 = \boxed{8}$

wrong $\displaystyle\lim_{x\to 0} \frac{\sin 8x}{x} \neq \lim_{x\to 0} \frac{8\sin x}{x}$

Sep 25-11:03 AM

Evaluate $\displaystyle\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4} = \frac{\sin(2-2)}{2^2-4} = \frac{\sin 0}{0} = \frac{0}{0}$

I.F.

$$= \lim_{x\to 2} \frac{\sin(x-2)}{(x+2)\cdot(x-2)} = \lim_{x\to 2} \frac{1}{x+2} \cdot \lim_{x\to 2} \frac{\sin(x-2)}{x-2}$$

$$= \frac{1}{2+2} \cdot \lim_{x\to 2} \frac{\sin(x-2)}{x-2}$$

$$= \frac{1}{4} \cdot 1 = \boxed{\frac{1}{4}}$$

Sep 25-11:08 AM

Evaluate $\lim_{x \to -2} \frac{\sin(x+2)}{x^3+8} = \frac{\sin 0}{0} = \frac{0}{0}$ I.F.

$$= \lim_{x \to -2} \frac{\sin(x+2)}{(x+2)(x^2-2x+4)} = \boxed{\lim_{x \to -2} \frac{\sin(x+2)}{x+2}} \cdot \boxed{\lim_{x \to -2} \frac{1}{x^2-2x+4}}$$

$$= 1 \cdot \frac{1}{(-2)^2-2(-2)+4}$$

$$= \boxed{\frac{1}{12}}$$

Sep 25-11:13 AM

Evaluate $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$ I.F.

$$= \lim_{x \to 0} \frac{\frac{3\sin 3x}{3x}}{\frac{5\sin 5x}{5x}} = \frac{3}{5} \lim_{x \to 0} \frac{\frac{\sin 3x}{3x}}{\frac{\sin 5x}{5x}}$$

$$= \frac{3}{5} \cdot \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 5x}{5x}} = \frac{3}{5} \cdot \frac{1}{1} = \boxed{\frac{3}{5}}$$

Sep 25-11:17 AM

Evaluate $\lim_{x\to 10} \frac{1-\cos(x-10)}{x^2-100} = \frac{1-\cos(10-10)}{10^2-100}$

$= \frac{1-\cos 0}{100-100} = \frac{1-1}{100-100}$

$= \frac{0}{0}$ I.F.

$= \lim_{x\to 10} \frac{1-\cos(x-10)}{(x+10)\cdot(x-10)}$

$h = x-10$

as $x \to 10$, $h \to 0$

$= \lim_{x\to 10} \frac{1}{x+10} \cdot \lim_{\substack{x\to 10 \\ h\to 0}} \frac{1-\cos(\overbrace{x-10}^{h})}{\underbrace{x-10}_{h}} =$

$= \frac{1}{10+10} \cdot \lim_{h\to 0} \frac{1-\cos h}{h} = \frac{1}{20}\cdot 0 = \boxed{0}$

$\emptyset \to$ Not Zero

Zero