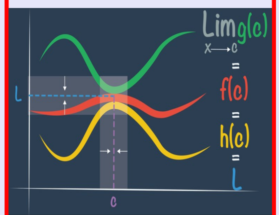
Math 261

Fall 2023

Lecture 13



Feb 19-8:47 AM

Given $f(x)=\frac{x}{x-4}$

1) Domain, $x \neq 4 \to (-\infty,4)\cup(4,\infty)$

2) Find $f(6)$ $\quad f(6)=\frac{6}{6-4}=\frac{6}{2}=3 \to (6,3)$

3) Find $f'(x)$ using definition of limit.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{x+h}{x+h-4}-\frac{x}{x-4}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{(x+h-4)}(x-4)\cdot\frac{x+h}{\cancel{x+h-4}}-(x+h-4)\cancel{(x-4)}\cdot\frac{x}{\cancel{x-4}}}{(x+h-4)(x-4)\cdot h}$$

$$=\lim_{h\to 0}\frac{(x-4)(x+h)-x(x+h-4)}{(x+h-4)(x-4)\cdot h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+\cancel{xh}-\cancel{4x}-4h-\cancel{x^2}-\cancel{xh}+\cancel{4x}}{(x+h-4)(x-4)\cdot h}$$

$$=\lim_{h\to 0}\frac{-4}{(x+h-4)(x-4)}=\boxed{\frac{-4}{(x-4)^2}} \leftarrow f'(x)$$

Sep 19-10:25 AM

4) Now find $f'(6)$

$$f'(x) = \frac{-4}{(x-4)^2} \qquad f'(6) = \frac{-4}{(6-4)^2} = \frac{-4}{2^2} = \boxed{-1}$$

5) Simplify

$$y - \underset{\sim}{f(6)} = f'(6)(x-6)$$

$$y - 3 = -1(x-6)$$

$$y - 3 = -x + 6$$

$$\boxed{y = -x + 9}$$

Sep 19-10:34 AM

Show $\underbrace{x^3 - 4x}\ + 1 = 0$ over $[1,2]$

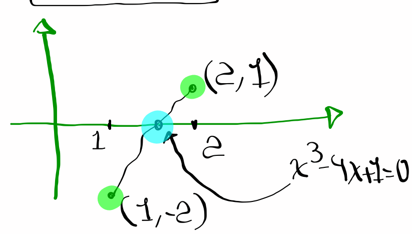
Let $f(x) = x^3 - 4x + 1$

Polynomial function

Always $\boxed{\text{continuous}}$ on $(-\infty, \infty)$

$f(1) = 1^3 - 4(1) + 1 = -2$

$f(2) = 2^3 - 4(2) + 1 = 1$



By intermediate value theorem, there is a number c in $(1,2)$ such that $f(c) = 0$

So $x^3 - 4x + 1 = 0$ has a solution in $(1,2)$.

Sep 19-10:38 AM

$$f(x)=\begin{cases} 7x-2, & x\le 1 \\ Kx^2, & x>1 \end{cases}$$

Find K such that $f(x)$ is continuous.

$f(x)$ is cont. at $x=a$ when $\lim_{x\to a} f(x) = f(a)$

$f(x)=7x-2$ is a linear function and is cont. everywhere.

$f(x)=Kx^2$ is a polynomial, and is cont. everywhere

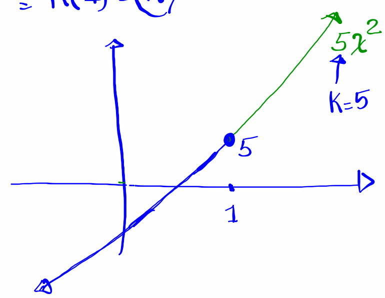
at $x=1$

$f(1)=7(1)-2=5$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(7x-2) = 7(1)-2 = 5$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} Kx^2 = K(1)^2 = K$

$\Rightarrow \boxed{K=5}$



Sep 19-10:46 AM

$$f(x)=\begin{cases} Kx^2 & \text{if } x\le 2 \\ 2x+K & \text{if } x>2 \end{cases}$$

Find K such that $f(x)$ is cont. at $x=2$

Both pieces are cont. every where

Kx^2 & $2x+K$

to eliminate this gap

$\lim_{x\to 2} f(x) = f(2)$

$f(2)=K(2)^2=4K$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} Kx^2 = 4K$

equal

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(2x+K) = 4+K$

$4K = 4+K$

$3K=4$

$\boxed{K=\frac{4}{3}}$

Sep 19-10:53 AM

For $\varepsilon = .001$, find a $\delta > 0$ such that $\lim_{x \to 4} x^2 = 16$. ✓

For $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 - 16| < \varepsilon$ " $|x - 4| < \delta$

$|(x+4)(x-4)| < \varepsilon$ $\delta = \frac{\varepsilon}{c}$

$|x+4||x-4| < \varepsilon$

If $|x+4| < C$, then $|x-4| < \frac{\varepsilon}{C}$

If $\delta \le 1$, then $|x-4| < \delta \le 1$

$|x-4| < 1$, $-1 < x - 4 < 1 \to 7 < x+4 < 9$

Add 8

So $|x+4| < 9$

$\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$

For $\varepsilon = .001$, $\delta = \min\left\{1, \frac{.001}{9}\right\} = \min\left\{1, \frac{1}{9000}\right\}$

$\boxed{\delta = \frac{1}{9000}}$

Sep 19-11:00 AM

Use ε and δ definition to prove $\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2$

$f(x) = \frac{1}{x}$ verify the limit

$L = 2$ $\lim_{x \to 1/2} \frac{1}{x} = \frac{1}{1/2} = 2$ ✓

$a = \frac{1}{2}$

For every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{1}{x} - 2\right| < \varepsilon$ " $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{1 - 2x}{x}\right| < \varepsilon$ " $\left|x - \frac{1}{2}\right| < \delta$

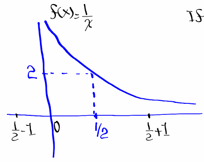
$\left|\frac{-2\left(x - \frac{1}{2}\right)}{x}\right| < \varepsilon$ " $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{-2}{x}\left(x - \frac{1}{2}\right)\right| < \varepsilon$ Pick $\delta = \frac{\varepsilon}{c}$

$\left|\frac{2}{x}\right|\left|x - \frac{1}{2}\right| < \varepsilon$

If $\left|\frac{2}{x}\right| < C$, then $C\left|x - \frac{1}{2}\right| < \varepsilon$, $\left|x - \frac{1}{2}\right| < \frac{\varepsilon}{C}$

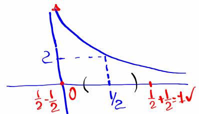
Sep 18-11:28 AM



If $\delta = 1$

$f(x)$ has a chance to be undefined on the left side

If $\delta = \frac{1}{2}$



Same challenge,

$f(x)$ ✓ on Right side

$f(x)$ has issue on left side

Pick δ no more than $\frac{1}{4}$ $\Rightarrow$ $\delta \le \frac{1}{4}$

$|x-\frac{1}{2}| < \delta \le \frac{1}{4}$ $\Rightarrow$ $|x-\frac{1}{2}| < \frac{1}{4}$

$-\frac{1}{4} < x-\frac{1}{2} < \frac{1}{4}$ $\Rightarrow$ $\frac{1}{4} < x < \frac{3}{8}$

Add $\frac{1}{2}$

Take reciprocal $\frac{4}{1} > \frac{1}{x} > \frac{8}{3}$

$4 > \frac{1}{x} > \frac{8}{3}$ $\Rightarrow$ $\frac{8}{3} < \frac{1}{x} < 4$

multiply by 2 $\frac{16}{3} < \frac{2}{x} < 8$

If $\frac{2}{x} < 8$, then $|\frac{2}{x}| < 8$ $C = 8$

$$\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{8}\right\}$$

If $\varepsilon = 1$, $\delta = \min\{\frac{1}{4}, \frac{1}{8}\} = \frac{1}{8}$

If $\varepsilon = 5$, $\delta = \min\{\frac{1}{4}, \frac{5}{8}\} = \min\{.25, .625\} = .25$

Sep 19-11:11 AM

For $\varepsilon = .05$, find $\delta > 0$ such that $\lim_{x\to 5} \frac{1}{x} = \frac{1}{5}$.

$f(x) = \frac{1}{x}$

$L = \frac{1}{5}$ $\quad \lim_{x\to 5} \frac{1}{x} = \frac{1}{5}$ ✓

$a = 5$

for $\varepsilon > 0$, there is $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$|\frac{1}{x} - \frac{1}{5}| < \varepsilon$ $\quad$ " $\quad |x-5| < \delta$

$|\frac{5-x}{5x}| < \varepsilon$

$|\frac{-(x-5)}{5x}| < \varepsilon$

$|\frac{-1}{5x}|\,|x-5| < \varepsilon$

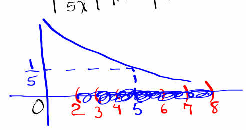
$\rightarrow |\frac{1}{5x}|\,|x-5| < \varepsilon$

if $|\frac{1}{5x}| < C$, then

$|x-5| < \frac{\varepsilon}{C}$

Pick $\delta = \frac{\varepsilon}{C}$

Pick $\delta \le 1$



$|x-5| < \delta \le 1$

$|x-5| < 1 \rightarrow -1 < x-5 < 1$ $\quad 4 < x < 6$

Add 5

multiply by 5 $\quad$ take reciprocal

$20 < 5x < 30$ $\quad \frac{1}{20} > \frac{1}{5x} > \frac{1}{30}$

$|\frac{1}{5x}| < \frac{1}{20} \leftarrow C$

$\delta = \min\left\{1, \frac{\varepsilon}{\frac{1}{20}}\right\} = \min\{1, 20\varepsilon\}$

for $\varepsilon = .05$

$\delta = \min\{1, 20(.05)\}$

$= \min\{1, 1\}$

$\boxed{1}$

Sep 19-11:24 AM