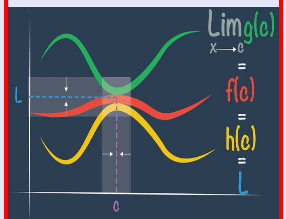
Math 261

Fall 2022

Lecture 13



Evaluate $\lim\limits_{x\to\infty}\sqrt{\dfrac{12x^3+7x-4}{3x^3-2x+8}} = \dfrac{\infty}{\infty}$ I.F.

Divide by highest power of x in the deno. $\Rightarrow x^3$

$$\lim_{x\to\infty}\sqrt{\frac{12x^3+7x-4}{3x^3-2x+8}} = \lim_{x\to\infty}\sqrt{\frac{\frac{12x^3}{x^3}+\frac{7x}{x^3}-\frac{4}{x^3}}{\frac{3x^3}{x^3}-\frac{2x}{x^3}+\frac{8}{x^3}}}$$

$$= \lim_{x\to\infty}\sqrt{\frac{12+\cancel{\frac{7}{x^2}}^{\to 0}-\cancel{\frac{4}{x^3}}^{\to 0}}{3-\cancel{\frac{2}{x^2}}^{\to 0}+\cancel{\frac{8}{x^3}}^{\to 0}}} = \lim_{x\to\infty}\sqrt{\frac{12}{3}} = \sqrt{4} = \boxed{2}$$

Find $\lim_{x\to\infty} \dfrac{\sqrt{9x^6+x}}{x^3+1} = \dfrac{\infty}{\infty}$ I.F.

$x^3 = \sqrt{x^6}$

Divide by x^3

$$\lim_{x\to\infty} \frac{\sqrt{9x^6+x}}{x^3+1} = \lim_{x\to\infty} \frac{\sqrt{\dfrac{9x^6+x}{x^6}}}{\dfrac{x^3}{x^3}+\dfrac{1}{x^3}} = \lim_{x\to\infty} \frac{\sqrt{9+\cancel{\dfrac{1}{x^5}}^{\to 0}}}{1+\cancel{\dfrac{1}{x^3}}^{\to 0}}$$

$$= \frac{\sqrt{9}}{1} = \boxed{3}$$

Evaluate $\lim_{x\to-\infty} \dfrac{\sqrt{9x^6+1}}{x^3+1} = \dfrac{\infty}{-\infty}$ I.F.

Divide by x^3, also $x^3 = \sqrt{x^6}$

as $x \to -\infty$

$-\infty$ ∞

to avoid this

Since $x\to-\infty$

$x^3 = -\sqrt{x^6}$

$$\lim_{x\to-\infty} \frac{\sqrt{9x^6+1}}{x^3+1} = \lim_{x\to-\infty} \frac{-\sqrt{\dfrac{9x^6+1}{x^6}}}{\dfrac{x^3+1}{x^3}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{9+\cancel{\dfrac{1}{x^6}}^{\to 0}}}{1+\cancel{\dfrac{1}{x^3}}^{\to 0}} = \frac{-\sqrt{9}}{1}$$

$$= \boxed{-3}$$

Find $\lim_{x\to\infty} \left[\sqrt{9x^2+x} - 3x\right] = \infty - \infty$ I.F.

$$\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right) = \lim_{x\to\infty} \frac{(\sqrt{9x^2+x} - 3x)(\sqrt{9x^2+x} + 3x)}{\sqrt{9x^2+x} + 3x}$$

$$= \lim_{x\to\infty} \frac{\cancel{9x^2} + x - \cancel{9x^2}}{\sqrt{9x^2+x} + 3x}$$

Divide by x

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\sqrt{\frac{9x^2}{x^2} + \frac{x}{x^2}} + \frac{3x}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{9 + \cancel{\frac{1}{x}}^{0}} + 3}$$

$$= \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

Find $\lim_{x\to\infty} x \sin\frac{1}{x} = \infty \cdot 0$ I.F.

Recall $\frac{1}{x} \to 0$ as $x\to\infty$

$\sin\frac{1}{x} \to \sin 0 = 0$

It will be nice if somehow we could have $\frac{\sin\frac{1}{x}}{\frac{1}{x}}$

$$\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \sin\frac{1}{x} \div \frac{1}{x} = \sin\frac{1}{x} \cdot x = x \sin\frac{1}{x}$$

$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$$

Let $\frac{1}{x} = h$

as $x\to\infty$
$h\to 0$

$$= \lim_{h\to 0} \frac{\sin h}{h} = \boxed{1}$$

Recall $\lim_{h\to 0} \frac{\sin h}{h} = 1$

$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

Find $\lim_{x\to\infty} \cos\frac{1}{x} = \lim_{h\to 0} \cos h = \cos 0$

$= \boxed{1}$

as $x\to\infty$ let $h = \frac{1}{x}$

$\frac{1}{x} \to 0$

Find $\lim_{x \to 0} \frac{\sin 3x}{5x^3 - 4x} = \frac{0}{0}$ I.F.

would be nice to get $\frac{\sin 3x}{3x}$ Since $\lim_{x \to 0} \frac{\sin 3x}{3x} = 1$

$$\lim_{x \to 0} \frac{\frac{\sin 3x}{3x}}{\frac{5x^3}{3x} - \frac{4x}{3x}} = \lim_{x \to 0} \frac{\cancel{\frac{\sin 3x}{3x}} \to 1}{\cancel{\frac{5x^2}{3}} \to 0 - \frac{4}{3}}$$

$$= \frac{1}{-\frac{4}{3}} = \boxed{\frac{-3}{4}}$$

Evaluate

$$\lim_{x \to 0} x \sin \frac{1}{x}$$

Since $\frac{1}{x}$ is undefined for $x = 0$, we cannot Plug it in.

$$-1 \le \sin A \le 1$$

$$-1 \le \sin \frac{1}{x} \le 1$$

Multiply by x $\quad -x \le x \sin \frac{1}{x} \le x$

if $x < 0$ $\quad x \le x \sin \frac{1}{x} \le -x$

$\lim_{x \to 0} x = 0$

$\lim_{x \to 0} -x = 0$

by S.T. $\lim_{x \to 0} x \sin \frac{1}{x} = \boxed{0}$

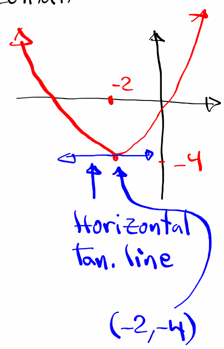
Find points on the graph of $f(x)=x^2+4x$ where tan. line is horizontal.

$f(x)=\underbrace{x^2+4x+4}-4$

$f(x)=(x+2)^2-4$

Parabola $y=a(x-h)^2+k$

Vertex $(-2,-4)$



using Calc. Method

$$m_{\text{tan. line}}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2+4(x+h)-x^2-4x}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{+4x}+4h\cancel{-x^2}\cancel{-4x}}{h}=\lim_{h\to 0}\frac{\cancel{h}(2x+h+4)}{\cancel{h}}$$

$$=\lim_{h\to 0}(2x+h+4)=2x+4$$

we want horizontal tan. line $\Rightarrow m=0$

$2x+4=0$ $\boxed{x=-2}$

Point $(-2,f(-2))\Rightarrow(-2,-4)$

For $\varepsilon>0$, find $\delta>0$ such that

$$\lim_{x\to\sqrt{2}}\frac{2}{x^2}=1$$ ✓

$f(x)=\frac{2}{x^2}$ $x\neq 0$ V.A.

$a=\sqrt{2}$

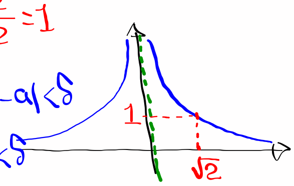
$L=1$

verify the limit

$$\lim_{x\to\sqrt{2}}\frac{2}{x^2}=\frac{2}{(\sqrt{2})^2}=\frac{2}{2}=1$$

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$\left|\frac{2}{x^2}-1\right|<\varepsilon$ " $|x-\sqrt{2}|<\delta$



Let's have $\delta\leq 1$

$\left|\frac{2-x^2}{x^2}\right|<\varepsilon$

$\left|\frac{-(x^2-2)}{x^2}\right|<\varepsilon$ $\Rightarrow$ $\frac{1}{x^2}|x^2-2|<\varepsilon$

$\frac{1}{x^2}|(x+\sqrt{2})(x-\sqrt{2})|<\varepsilon$

$\underbrace{\frac{|x+\sqrt{2}|}{x^2}}_{\text{Bound}}\underbrace{|x-\sqrt{2}|}_{\text{keep}}<\varepsilon$

So we wanted $\delta\leq 1$

$|x-\sqrt{2}|<\delta\leq 1$

$|x-\sqrt{2}|<1$

$-1<x-\sqrt{2}<1$ $\Rightarrow$ $-1+\sqrt{2}<x<1+\sqrt{2}<3$

So $x<3$

What about $x+\sqrt{2}$? $\quad x+\sqrt{2} < 3+\sqrt{2} < 3+2 < 5$

$x+\sqrt{2} < 5$

What about x^2? $\quad$ If $x<3$, $x^2<9$

Now $\dfrac{|x+\sqrt{2}|}{x^2} < \dfrac{5}{9} < 1$

$\dfrac{|x+\sqrt{2}|}{x^2}\,|x-\sqrt{2}| < 1\,|x-\sqrt{2}| < \varepsilon$

Pick $\delta = \varepsilon$

Keep in mind $\delta \le 1$

$\delta = \min\{1, \varepsilon\}$

For $\varepsilon = .1$

$L = 1$

$L+\varepsilon = 1.1$

$L-\varepsilon = .9$

$f(x) = 1.1$

$\dfrac{2}{x^2} = 1.1$

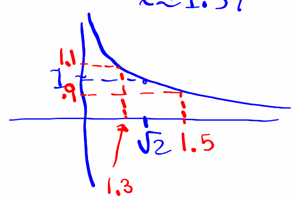
$x^2 = \dfrac{2}{1.1}$

$x \approx 1.3$

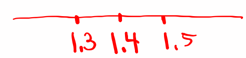
$f(x) = .9$

$\dfrac{2}{x^2} = .9$

$x^2 = \dfrac{2}{.9}$ $\quad x \approx 1.5$



$\sqrt{2} \approx 1.4$



$\delta = .1$