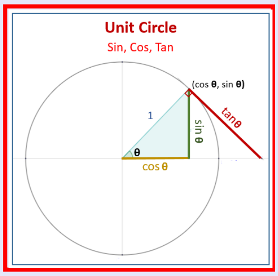
Math 241

Winter 2023

Lecture 9



Feb 19-8:47 AM

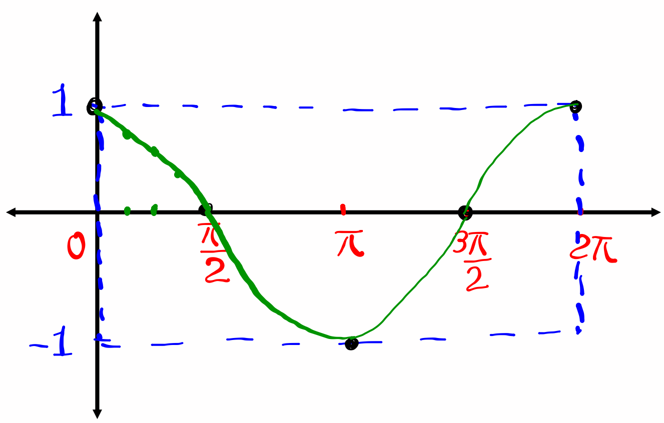
x	0° 0	30° $\pi/6$	45° $\pi/4$	60° $\pi/3$	90° $\pi/2$	180° π	270° $3\pi/2$	360° 2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1

$y = \cos x$

one period

$0 \le x \le 2\pi$

$-1 \le \cos x \le 1$



Jan 18-7:01 AM

Graph $y = \cos\left(x - \frac{\pi}{2}\right)$ Amplitude $= 1$

$0 \le x - \frac{\pi}{2} \le 2\pi$

Multiply by 2

$0 \le 2x - \pi \le 4\pi$

Add π

$\pi \le 2x \le 5\pi$

Divide by 2

$\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

x	$\frac{\pi}{2}$	$\frac{2\pi}{2}$ (π)	$\frac{3\pi}{2}$	$\frac{4\pi}{2}$ (2π)	$\frac{5\pi}{2}$
$x - \pi/2$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\left(x - \frac{\pi}{2}\right)$	1	0	-1	0	1

No vertical shift

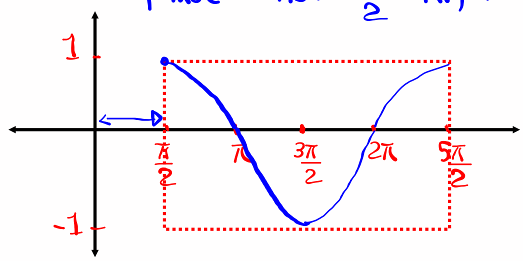
phase shift $\frac{\pi}{2}$ Right

$\frac{\pi}{2} + \frac{5\pi}{2} = \frac{6\pi}{2} = 3\pi$

Midpoint $\frac{3\pi}{2}$

$\frac{\pi}{2} + \frac{3\pi}{2} = \frac{4\pi}{2} = 2\pi$

Midpoint $\frac{2\pi}{2}$



Jan 18-7:08 AM

Graph $y = -\cos\left(\frac{1}{2}x\right)$ Amplitude $= 1$

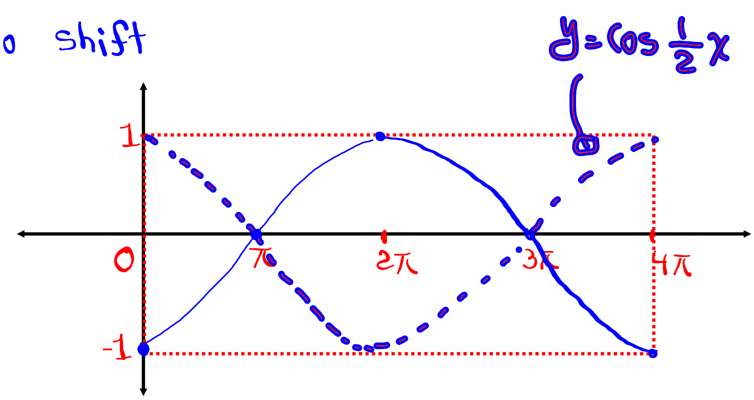
$0 \le \frac{1}{2}x \le 2\pi$

Multiply by 2

$0 \le x \le 4\pi$

x	0	π	2π	3π	4π
$\frac{1}{2}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\frac{1}{2}x$	1	0	-1	0	1
$-\cos\frac{1}{2}x$	-1	0	1	0	-1

No shift



Jan 18-7:20 AM

Graph $y = 2\cos\left(\frac{1}{2}x + \pi\right)$

Amplitude $= 2$

$0 \le \frac{1}{2}x + \pi \le 2\pi$

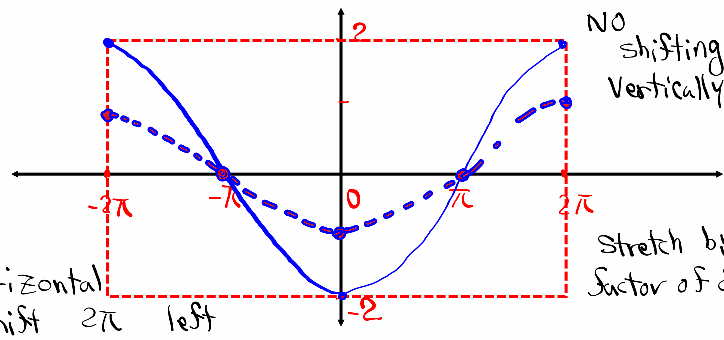
Subtract π

$-\pi \le \frac{1}{2}x \le \pi$

Multiply by 2

$-2\pi \le x \le 2\pi$

x	-2π	$-\pi$	0	π	2π
$\frac{1}{2}x+\pi$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos(\frac{1}{2}x+\pi)$	1	0	-1	0	1
$2\cos(\frac{1}{2}x+\pi)$	2	0	-2	0	2



NO shifting Vertically

Stretch by factor of 2.

Horizontal shift 2π left

Jan 18-7:31 AM

Graph $y = -2 + \cos(2x)$

Vertical shift 2 down

Amplitue $= 1$

NO phase shift

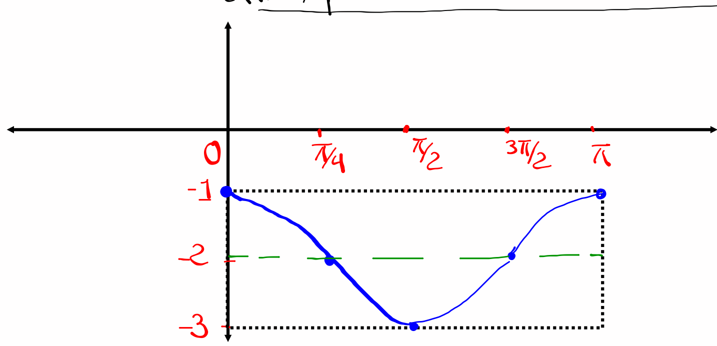
Compress by factor of 2.

$0 \le 2x \le 2\pi$

Divide by

$0 \le x \le \pi$

	$0°$	$45°$	$90°$	$135°$	$180°$
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$2x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos 2x$	1	0	-1	0	1
$-2+\cos 2x$	-1	-2	-3	-2	-1



Jan 18-7:43 AM

$y = 1 + \frac{1}{2}\cos(2x - 2\pi)$

Vertical shift 1 up

Amplitude $= \frac{1}{2}$

x	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$2x-2\pi$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos(2x-2\pi)$	1	0	-1	0	1
$\frac{1}{2}\cos(2x-2\pi)$	$\frac{1}{2}$	0	$\frac{-1}{2}$	0	$\frac{1}{2}$
$1+\frac{1}{2}\cos(2x-2\pi)$	$\frac{3}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$

$0 \le 2x - 2\pi \le 2\pi$

$2\pi \le 2x \le 4\pi$

$\pi \le x \le 2\pi$

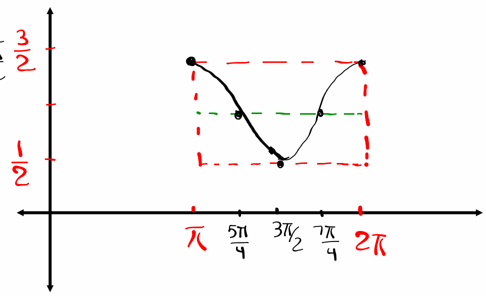
$\pi + \frac{3\pi}{2} = \frac{2\pi}{2} + \frac{3\pi}{2} = \frac{5\pi}{2}$

$\frac{5\pi}{2} \div 2 = \frac{5\pi}{4}$

$270^\circ + 360^\circ = 630^\circ$

$630^\circ = 315^\circ$

$315^\circ = \overset{7\ \cancel{63}}{\cancel{315}} \cdot \frac{\pi}{\underset{\cancel{36}\ 4}{\cancel{180}}}$



Jan 18-7:54 AM

Sum & Difference Formulas:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Jan 18-8:28 AM

Find exact Value of $\tan 15^\circ$.

Hint: $15^\circ = 45^\circ - 30^\circ$

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \underbrace{\tan 45^\circ}_{1} \tan 30^\circ}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = \frac{(3-\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}$$

LCD = 3 Rationalize the deno.

$$= \frac{9 - 3\sqrt{3} - 3\sqrt{3} + \sqrt{9}}{9 - \cancel{3\sqrt{3}} \cancel{+3\sqrt{3}} - \sqrt{9}} = \frac{12 - 6\sqrt{3}}{6} = \frac{\cancel{6}(2-\sqrt{3})}{\cancel{6}}$$

$$= \boxed{2-\sqrt{3}}$$

Jan 18-8:32 AM

Find exact value for $\cos 105^\circ$

Hint: $105^\circ = 60^\circ + 45^\circ$

$$\cos 105^\circ = \cos(60^\circ + 45^\circ)$$

$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2} - \sqrt{6}}{4}}$$

Jan 18-8:38 AM

Verify $\sin(180° - \theta) = \sin\theta$ ✓

$\sin(180° - \theta) = \sin 180° \cos\theta - \cos 180° \sin\theta$ ($\sin 180° \to 0$, $\cos 180° \to -1$)

$= 0 \cdot \cos\theta - (-1)\cdot\sin\theta$

$= \sin\theta$ ✓

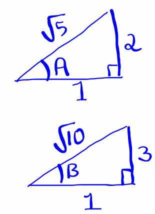
$\cos A = \frac{1}{\sqrt{5}}$

$\sec A = \sqrt{5}$, A is in QI

$\sec B = \sqrt{10}$, B is in QIV

Find $\sec(A-B)$

Hint: Find $\cos(A-B)$



$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} + \frac{2}{\sqrt{5}} \cdot \frac{-3}{\sqrt{10}} = \frac{1}{\sqrt{50}} - \frac{6}{\sqrt{50}}$

$= \frac{1-6}{\sqrt{50}} = \frac{-5}{\sqrt{50}}$

$\sec(A-B) = -\frac{\sqrt{50}}{5} = -\frac{\sqrt{25}\sqrt{2}}{5} = -\frac{5\sqrt{2}}{5} = \boxed{-\sqrt{2}}$

Jan 18-8:41 AM

Verify $\frac{\cos(A+B)}{\sin A \cos B} = \cot A - \tan B$ ✓

Hint: Expand $\cos(A+B)$

$\frac{\cos(A+B)}{\sin A \cos B} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B}$

$= \frac{\cos A \cancel{\cos B}}{\sin A \cancel{\cos B}} - \frac{\cancel{\sin A} \sin B}{\cancel{\sin A} \cos B}$

$= \frac{\cos A}{\sin A} - \frac{\sin B}{\cos B} = \boxed{\cot A - \tan B}$ ✓

Jan 18-8:50 AM

Simplify

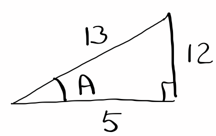
$$\sin(45^\circ + x) + \sin(45^\circ - x)$$

$$= \underbrace{\sin 45^\circ \cos x + \cancel{\cos 45^\circ \sin x}} + \underbrace{\sin 45^\circ \cos x - \cancel{\cos 45^\circ \sin x}}$$

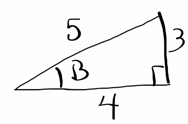
$$= 2 \sin 45^\circ \cos x = \cancel{2} \cdot \frac{\sqrt{2}}{\cancel{2}} \cos x = \boxed{\sqrt{2} \cos x}$$

Jan 18-8:55 AM

$\cos A = \frac{-5}{13}$, A in QIII



$\sin B = \frac{3}{5}$, B is QI



Find $\tan(A+B)$

$$= \frac{\overset{+}{\tan A} + \overset{+}{\tan B}}{1 - \overset{+}{\tan A}\,\overset{+}{\tan B}} = \frac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{12}{5} \cdot \frac{3}{4}}$$

LCD = 20

$$= \frac{\overset{4}{\cancel{20}} \cdot \frac{12}{\cancel{5}} + \overset{5}{\cancel{20}} \cdot \frac{3}{\cancel{4}}}{20 \cdot 1 - \cancel{20} \cdot \frac{12}{\cancel{5}} \cdot \frac{3}{\cancel{4}}} = \frac{48 + 15}{20 - 36} = \frac{63}{-16}$$

$$= \boxed{-\frac{63}{16}}$$

Jan 18-9:00 AM

Double - Angle Formula

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Suppose $B = A$

$\sin(A+A) = \sin A \cos A + \cos A \sin A$

$$\boxed{\sin 2A = 2 \sin A \cos A}$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Replace B by A

$\cos(A+A) = \cos A \cos A - \sin A \sin A$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$$

$\sin^2 A = 1 - \cos^2 A$

$\cos 2A = \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$$\boxed{\cos 2A = 2\cos^2 A - 1}$$

we can also show that

$$\boxed{\cos 2A = 1 - 2\sin^2 A}$$

Jan 18-9:06 AM

Find a formula in terms of $\tan A$ for $\tan 2A$.

Hint: work with $\tan(A+B)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

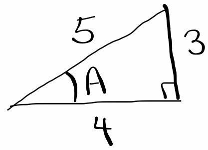
Replace B by A

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} \Rightarrow$$

$$\boxed{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

Jan 18-9:12 AM

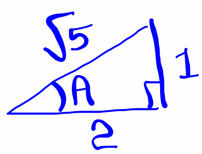
$\sin A = \frac{3}{5}$, A is in QII, find $\sin 2A$



$$\sin 2A = 2 \sin A \cos A$$
$$= 2 \cdot \frac{3}{5} \cdot \frac{-4}{5}$$
$$= \boxed{\frac{-24}{25}}$$

Jan 18-9:16 AM

$\sin A = \frac{1}{\sqrt{5}}$, find $\cos 2A$



$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2$$
$$= \frac{4}{5} - \frac{1}{5} = \boxed{\frac{3}{5}} ✓$$

$$\cos 2A = 2\cos^2 A - 1$$
$$= 2\left(\frac{2}{\sqrt{5}}\right)^2 - 1$$
$$= \frac{8}{5} - \frac{5}{5} = \boxed{\frac{3}{5}} ✓$$

$$\cos 2A = 1 - 2\sin^2 A$$
$$= 1 - 2\left(\frac{1}{\sqrt{5}}\right)^2$$
$$= 1 - \frac{2}{5} = \frac{5}{5} - \frac{2}{5}$$
$$= \boxed{\frac{3}{5}} ✓$$

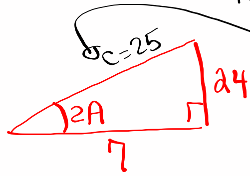
Jan 18-9:18 AM

$\tan A = \frac{3}{4}$, A is in QIII, find $\tan 2A$.

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\cdot\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1-\frac{9}{16}}$$

LCD = 16

$$= \frac{\overset{8}{\cancel{16}}\cdot\frac{3}{\cancel{2}}}{16\cdot 1-\cancel{16}\cdot\frac{9}{\cancel{16}}} = \frac{24}{16-9} = \boxed{\frac{24}{7}} > 0$$



$c^2 = 7^2 + 24^2$

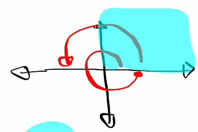
$= 625$

$c = 25$

$\cos 2A = \frac{7}{25}$

$180° < A < 270°$ QIII

$360° < 2A < 540°$



2A could be in QI or QII

Jan 18-9:24 AM

Verify $\sin 2x = \frac{2\cot x}{1+\cot^2 x}$ ✓

$$\frac{2\cot x}{1+\cot^2 x} = \frac{2\cdot\frac{\cos x}{\sin x}}{1+\frac{\cos^2 x}{\sin^2 x}} = \frac{2\cdot \sin^{\cancel{2}} x\cdot\frac{\cos x}{\cancel{\sin x}}}{\sin^2 x\cdot 1+\cancel{\sin^2 x}\cdot\frac{\cos^2 x}{\cancel{\sin^2 x}}}$$

LCD = $\sin^2 x$

$$= \frac{2\sin x\cos x}{\sin^2 x+\cos^2 x}$$

$$= \frac{2\sin x\cos x}{1}$$

$$= 2\sin x\cos x$$

$$= \sin 2x$$ ✓

Jan 18-9:31 AM

Verify $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$

Hint:

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$\sin 2x = 2\sin x \cos x$

$$\frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x}$$

$$= \frac{\cancel{1}-\cancel{1}+\cancel{2}\sin^{\cancel{2}} x}{\cancel{2}\cancel{\sin x}\cos x}$$

$$= \frac{\sin x}{\cos x} = \boxed{\tan x}$$

$$\frac{1-\cos 2x}{\sin 2x} = \frac{1-(2\cos^2 x-1)}{2\sin x\cos x}$$

$$= \frac{1-2\cos^2 x+1}{2\sin x\cos x}$$

$$= \frac{2-2\cos^2 x}{2\sin x\cos x}$$

$$= \frac{\cancel{2}(1-\cos^2 x)}{\cancel{2}\sin x\cos x}$$

$$= \frac{\sin^{\cancel{2}} x}{\cancel{\sin x}\cos x}$$

$$= \frac{\sin x}{\cos x} = \boxed{\tan x}$$

Jan 18-9:35 AM

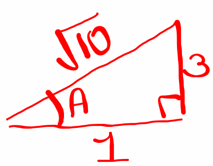
$\tan A = 3$, A is in QI Find $\csc 2A$.

$\rightarrow A = \tan^{-1} 3$

$A \approx 72^\circ$

Hint: Find $\sin 2A$

$\sin 2A = 2\sin A\cos A$



$$= 2 \cdot \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} = \frac{6}{10} = \frac{3}{5}$$

$\underbrace{\sqrt{10}\ \sqrt{10}}_{\sqrt{100}}$

$$\boxed{\csc 2A = \frac{5}{3}}$$

$2A \approx 144^\circ$

$2A$ is in QII

Jan 18-9:45 AM

Verify $\cos 4A = \cos^4 A - 6\cos^2 A \sin^2 A + \sin^4 A$ ✓

Hint

$4A = 2(2A)$

$\cos 4A = \cos[2(2A)]$

$= \cos^2 2A - \sin^2 2A$

$= [\cos^2 A - \sin^2 A]^2 - [2\sin A \cos A]^2$

$= (\cos^2 A - \sin^2 A)(\cos^2 A - \sin^2 A) - 4\sin^2 A \cos^2 A$

$= \cos^4 A - \cos^2 A \sin^2 A - \sin^2 A \cos^2 A + \sin^4 A - 4\sin^2 A \cos^2 A$

$= \cos^4 A - 6\cos^2 A \sin^2 A + \sin^4 A$

✓

Jan 18-9:51 AM

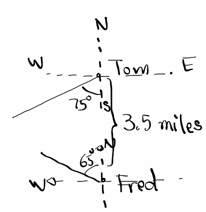
Fred is due South of Tom.

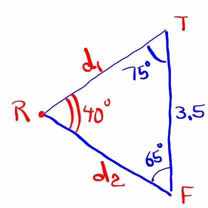
They are 3.5 miles apart

Tom estimates the bearing of a rocket to be S 75° W.

Fred estimates N 65° W.

How far are they from the rocket?





using law of Sines

$$\frac{\sin 65^\circ}{d_1} = \frac{\sin 40^\circ}{3.5} = \frac{\sin 75^\circ}{d_2}$$

$$d_1 = \frac{3.5 \sin 65^\circ}{\sin 40^\circ} \approx \boxed{4.9 \text{ miles}}$$

$$\frac{\sin 40^\circ}{3.5} = \frac{\sin 75^\circ}{d_2}$$

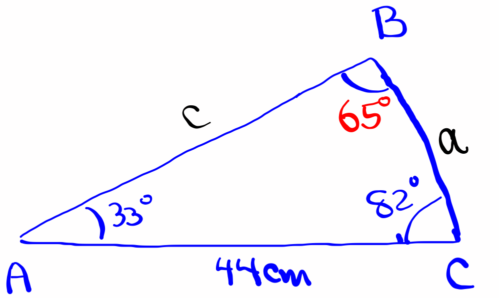
$$d_2 = \frac{3.5 \sin 75^\circ}{\sin 40^\circ}$$

$d_2 \approx 5.3$ miles

Jan 18-10:27 AM

Consider $\triangle ABC$ with $A = 33°$, $C = 82°$, $b = 44\,cm$

find B, find c.



$B = 180° - (33° + 82°)$

$= 180° - 115°$

$= 65°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 33°}{a} = \boxed{\frac{\sin 65°}{44} = \frac{\sin 82°}{c}}$$

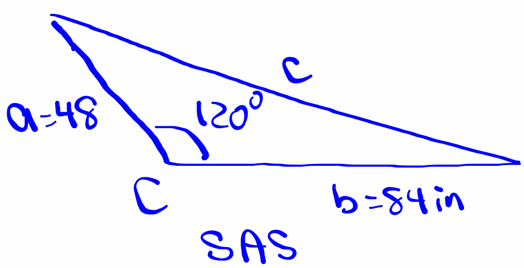
$$c = \frac{44 \sin 82°}{\sin 65°} = 48.07....$$

$$\boxed{c \approx 48\,cm}$$

Jan 18-10:37 AM

$a = 48$ in., $b = 84$ in., and $C = 120°$ in $\triangle ABC$

Find side c.



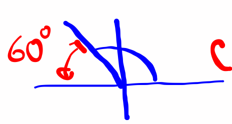
Law of Cosines

$c^2 = a^2 + b^2 - 2ab\cos C$

$= 48^2 + 84^2 - 2 \cdot 48 \cdot 84 \cdot \cos 120°$

$= 2304 + 7056 -$

$\cancel{2} \cdot 48 \cdot 84 \cdot \frac{-1}{\cancel{2}}$



$\cos 60° = \frac{1}{2}$

$c^2 = 2304 + 7056 + 4032$

$c^2 = 13392$

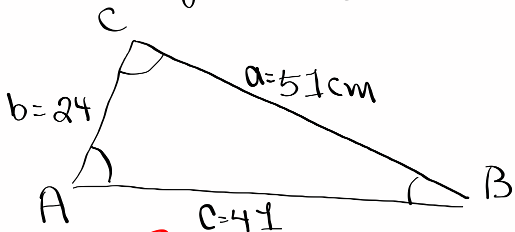
$c = \sqrt{13392} = 115.723...$

$c \approx 116$ in.

Jan 18-10:42 AM

In $\triangle ABC$, $a = 51\text{cm}$, $b = 24\text{cm}$, $c = 41\text{cm}$

Find the largest angle.



$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$2bc\cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{24^2 + 41^2 - 51^2}{2(24)(41)}$$

$$\cos A = \frac{-344}{1968}$$

$$A = \cos^{-1}(-.175)$$

$$\boxed{A \approx 100^\circ}$$

Jan 18-10:48 AM

Two planes leave the same airport at the same time.

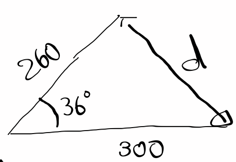
One plane has a speed of 130mph while the other plane has a speed of 150 mph.

Angle between their path is 36°.

How far apart are they after 2 hrs?

$d = 2 \cdot 130 = 260$ miles

$d = 2(150) = 300$ miles



Law of Cosines

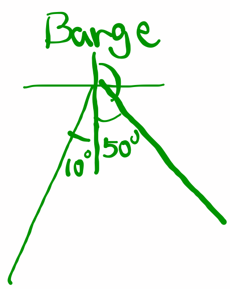
$$d^2 = 260^2 + 300^2 - 2 \cdot 260 \cdot 300 \cdot \cos 36^\circ$$

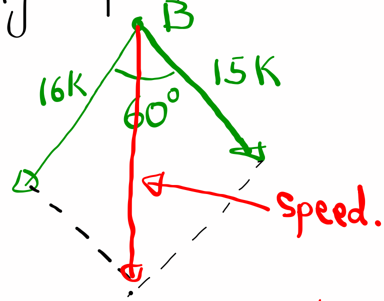
$$d^2 = 31393.34888 \qquad d = \sqrt{31393.34888}$$

$d \approx 177$ miles apart.

Jan 18-10:55 AM

A barge is pulled by two tugboats.
First tugboat travels at 15 knots with bearing of 130°.
Second tugboat travels at 16 knots with bearing of 190°. Find the resulting speed and direction of the barge.



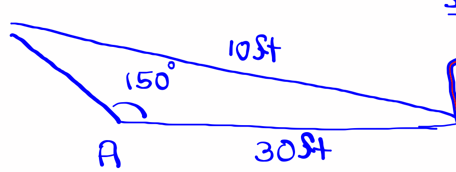


wait for vectors.

Jan 18-11:02 AM

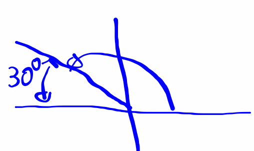
Solve triangle ABC if

$A = 150°$, $b = 30$ ft, $a = 10$ ft.



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\boxed{\frac{\sin 150°}{10} = \frac{\sin B}{30}} = \frac{\sin C}{c}$$



$\sin 30° = \frac{1}{2}$

$$10 \sin B = 30 \sin 150°$$

$$10 \sin B = 30 \cdot \frac{1}{2}$$

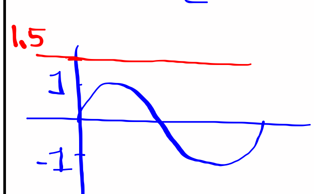
$$10 \sin B = 15$$

$$\sin B = 1.5$$

NO Solution

$B = \sin^{-1}(1.5)$

Error

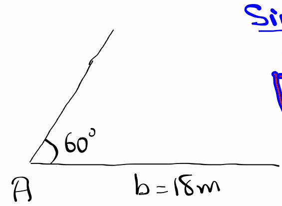


NO Such triangle

Jan 18-11:12 AM

$A = 60°$, $b = 18m$, $a = 16m$, Solve ΔABC.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

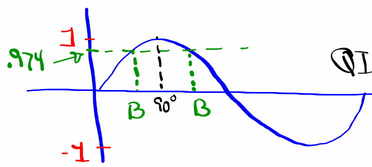


$$\boxed{\frac{\sin 60°}{16} = \frac{\sin B}{18}} = \frac{\sin C}{c}$$

$$\sin B = \frac{18 \cdot \sin 60°}{16} = .974$$

$RA = \sin^{-1}(.974)$

$RA \approx 77°$



QI $B = 77°$

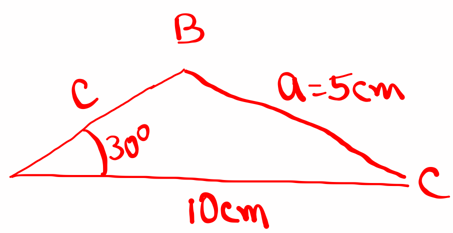
QII $B = 180° - 77° = 103°$

$A = 60°$
$B = 77°$
$C = 43°$
$c =$

$A = 60°$
$B = 103°$
$C = 17°$
$c =$

Jan 18-11:18 AM

Solve ΔABC if $A = 30°$, $b = 10cm$, $a = 5cm$.

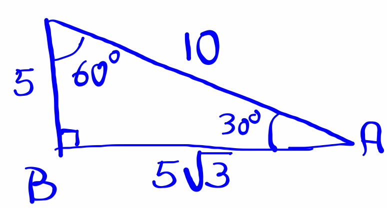


$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\boxed{\frac{\sin 30°}{5} = \frac{\sin B}{10}} = \frac{\sin C}{c}$$

$$\sin B = \frac{10 \cdot \sin 30°}{5} = \frac{10 \cdot \frac{1}{2}}{5} = \frac{5}{5} = 1$$

$B = 90°$ $C = 60°$



Jan 18-11:26 AM