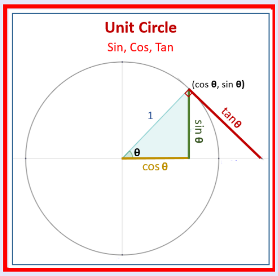
Math 241

Winter 2023

Lecture 17



Feb 19-8:47 AM

Let's review complex numbers

Complex number $\Rightarrow$ $\underset{\text{Real Part}}{a} + \underset{\text{Imaginary Part}}{b}i$

$i = \sqrt{-1}$

$i^2 = -1$

ex: $-2+8i$
- -2 Re. Part
- 8 Im. Part

$3-4i$
- Re. Part 3
- Im. Part -4

$-5i$
- Re. Part 0
- Im. Part -5

Absolute value of $a+bi$

$$|a+bi| = \sqrt{a^2+b^2}$$

ex: Find Abs. Value of $-8-6i$.

Re. Part -8

Im. Part -6

$$|-8-6i| = \sqrt{(-8)^2+(-6)^2} = 10$$

Given $Z = 6-6i$

Re. Part $= 6$

Im. Part $= -6$

$$|Z| = \sqrt{(6)^2+(-6)^2} = \sqrt{72}$$
$$= 6\sqrt{2}$$

Feb 1-7:03 AM

How to Plot a complex number:

Plot $5+2i$

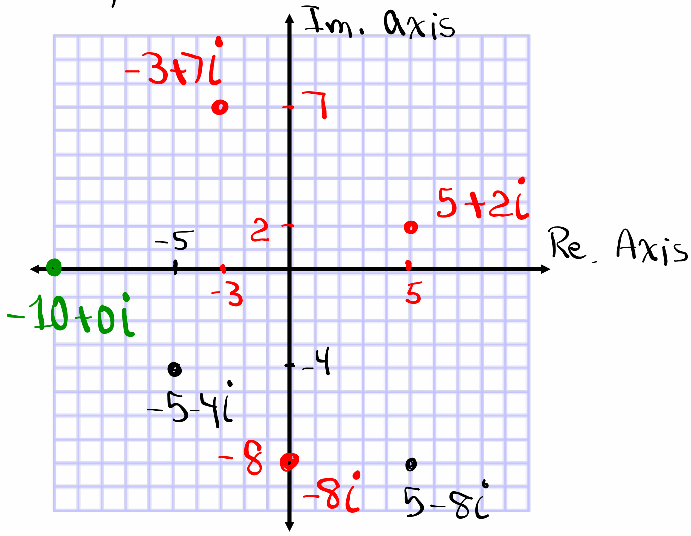
Plot $-3+7i$

Plot $-8i$

Plot $-5-4i$

Plot $5-8i$

Plot $-10+0i$



Feb 1-7:10 AM

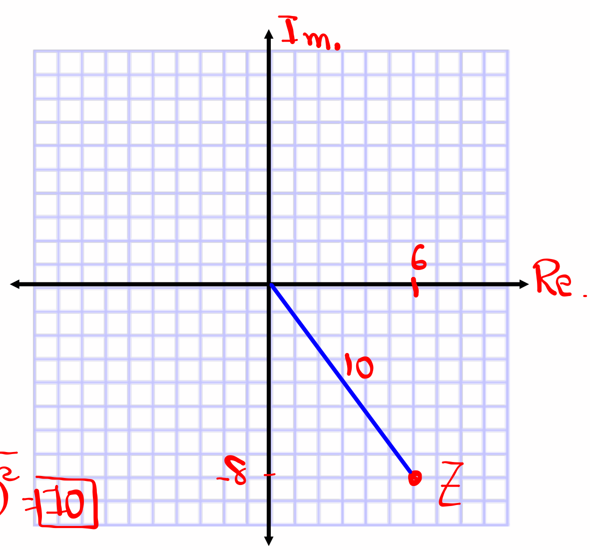
Given $Z = 6 - 8i$

1) Re. Part 6

2) Im. Part -8

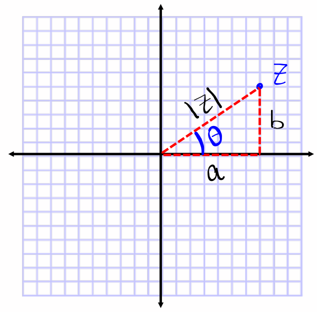
3) Plot Z

4) Find $|Z| = \sqrt{6^2 + (-8)^2} = \boxed{10}$



Feb 1-7:15 AM

$Z = a + bi$



$\cos\theta = \dfrac{a}{|Z|}$

$\sin\theta = \dfrac{b}{|Z|}$

$a = |Z| \cos\theta$

$b = |Z| \sin\theta$

If we let $r = |Z|$

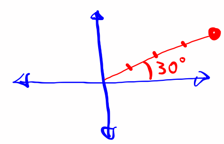
$Z = a + bi = r\cos\theta + r\sin\theta\, i$

$= r\left[\cos\theta + \sin\theta\, i\right]$

$= r\left[\cos\theta + i\sin\theta\right] = r\,\text{Cis}\,\theta$

θ is the angle made with Re. Axis > 0

Plot $Z = 4\ \text{Cis}\ 30°$



Feb 1-7:19 AM

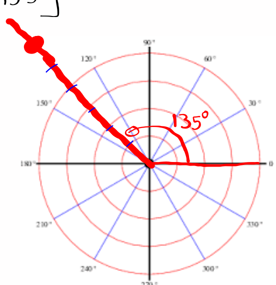
$Z = 6\ \text{Cis}\ 135°$

$= 6\left[\cos 135° + i\sin 135°\right]$

r $\quad$ $\theta = 135°$

$Z = 6\left[-\dfrac{\sqrt{2}}{2} + i \cdot \dfrac{\sqrt{2}}{2}\right]$

$\boxed{Z = -3\sqrt{2} + 3\sqrt{2}\, i}$



Given $Z = 4\ \text{cis}\ 210°$

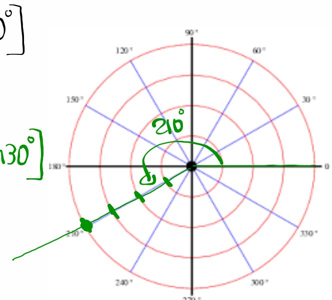
$Z = 4\left[\cos 210° + i\sin 210°\right]$

$r = 4$

$= 4\left[-\cos 30° + i \cdot -\sin 30°\right]$

$= 4\left[-\dfrac{\sqrt{3}}{2} - i \cdot \dfrac{1}{2}\right]$

$= \boxed{-2\sqrt{3} - 2i}$



Feb 1-7:25 AM

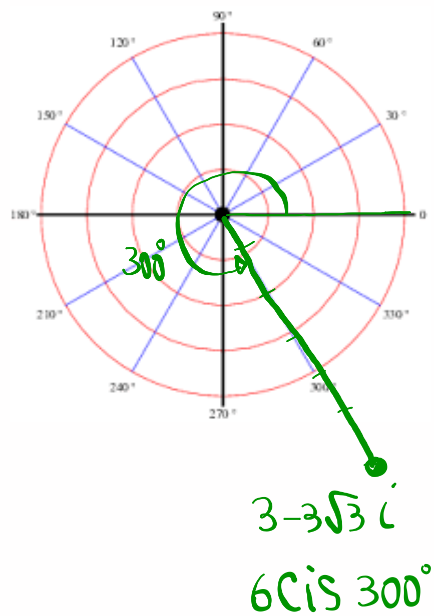
$Z = 6\,\text{cis}\,300°$

$= 6\left[\cos 300° + i\sin 300°\right]$

$= 6\left[\cos 60° + i \cdot -\sin 60°\right]$

$= 6\left[\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}\right]$

$= \boxed{3 - 3\sqrt{3}\,i}$



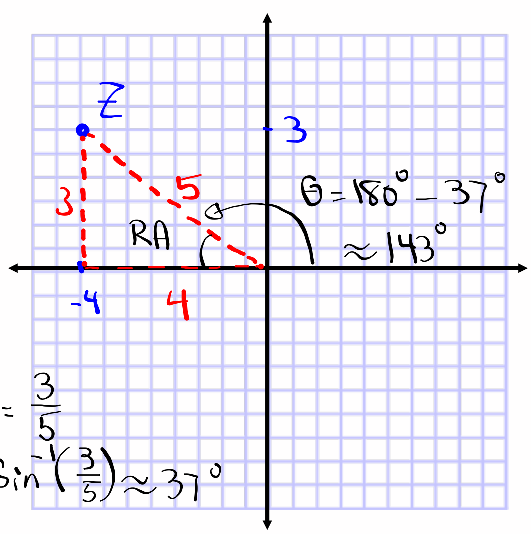
Feb 1-7:31 AM

$Z = -4 + 3i$

1) what Quadrant does Z belong to? QII

2) $|Z| = \sqrt{(-4)^2 + 3^2} = 5$

3) Plot Z



$\sin RA = \frac{3}{5}$

$RA = \sin^{-1}\left(\frac{3}{5}\right) \approx 37°$

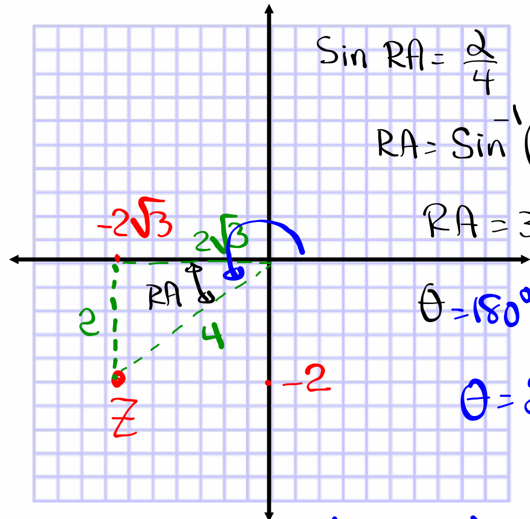
$Z = r(\cos\theta + i\sin\theta)$

$= 5(\cos 143° + i\sin 143°) = 5\,\text{cis}\,143°$

Feb 1-7:34 AM

$Z = -2\sqrt{3} - 2i$

1) Plot Z



$\sin RA = \frac{2}{4}$

$RA = \sin^{-1}\left(\frac{1}{2}\right)$

$RA = 30^\circ$

$\theta = 180^\circ + 30^\circ$

$\theta = 210^\circ$

2) $|Z|$

$|Z| = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$

$= \sqrt{12 + 4} = 4$

$Z = r(\cos\theta + i\sin\theta)$

$= 4[\cos 210^\circ + i\sin 210^\circ]$

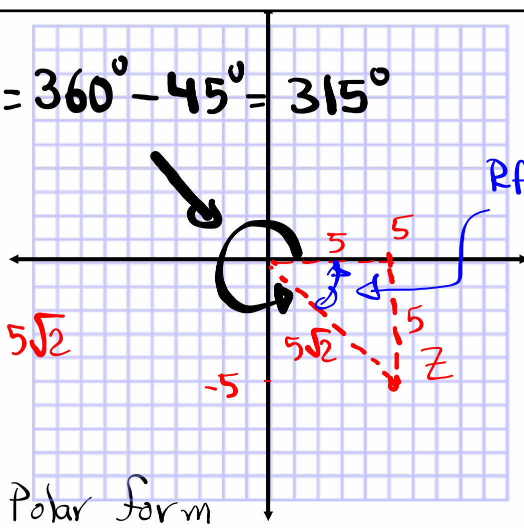
$= 4 \text{ cis } 210^\circ$

Feb 1-7:39 AM

$Z = 5 - 5i$

$\theta = 360^\circ - 45^\circ = 315^\circ$

1) Plot Z



2) $r = |Z| = \sqrt{5^2 + (-5)^2} = 5\sqrt{2}$

3) Express Z in Polar form

$Z = r(\cos\theta + i\sin\theta)$

$= 5\sqrt{2}[\cos 315^\circ + i\sin 315^\circ] = 5\sqrt{2} \text{ cis } 315^\circ$

Feb 1-7:44 AM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

$$z_1 z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right) \text{ and}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right) \quad r_2 \neq 0$$

Suppose $z_1 = 10(\cos 25^\circ + i\sin 25^\circ)$ and

$z_2 = 2(\cos 10^\circ + i\sin 10^\circ)$

$$z_1 z_2 = 10\cdot 2\left[\cos(25^\circ+10^\circ) + i\sin(25^\circ+10^\circ)\right]$$
$$= 20\left[\cos 35^\circ + i\sin 35^\circ\right] = 20\,\text{Cis}\,35^\circ$$

$$\frac{z_1}{z_2} = \frac{10}{2}\left[\cos(25^\circ-10^\circ) + i\sin(25^\circ-10^\circ)\right]$$
$$= 5\left[\cos 15^\circ + i\sin 15^\circ\right] = 5\,\text{Cis}\,15^\circ$$

Feb 1-8:24 AM

$z_1 = 6\,\text{Cis}\,60^\circ$, $z_2 = 2\,\text{Cis}\,30^\circ$

$$z_1 z_2 = 6\cdot 2\,\text{Cis}\,\underset{60^\circ+30^\circ}{90^\circ} = 12\,\text{Cis}\,90^\circ$$
$$= 12\left[\cancel{\cos 90^\circ}^{\,0} + i\,\cancel{\sin 90^\circ}^{\,1}\right]$$
$$= 12i$$

$$\frac{z_1}{z_2} = \frac{6}{2}\,\text{Cis}\,\underset{60^\circ-30^\circ}{30^\circ} = 3\,\text{Cis}\,30^\circ$$
$$= 3\left[\cos 30^\circ + i\sin 30^\circ\right]$$
$$= 3\left[\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right] = \boxed{\frac{3\sqrt{3}}{2} + \frac{3}{2}i}$$

Feb 1-8:30 AM

Suppose $Z_1 = 8\ \text{Cis}\ \frac{2\pi}{3}$, $Z_2 = 6\ \text{Cis}\ \frac{\pi}{4}$

$Z_1 = 8\ \text{Cis}\ 120°$, $Z_2 = 6\ \text{Cis}\ 45°$

$$Z_1 Z_2 = 8 \cdot 6\ \text{Cis}(120° + 45°)$$
$$= 48\ \text{Cis}\ 165°$$

$$\frac{Z_1}{Z_2} = \frac{8}{6}\ \text{Cis}(120° - 45°) = \frac{4}{3}\ \text{Cis}\ 75°$$

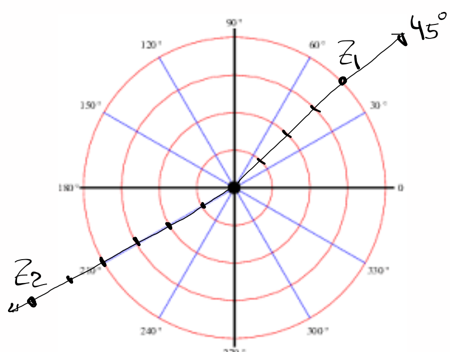
Feb 1-8:34 AM

$Z_1 = 4\ \text{Cis}\ 45°$ $Z_2 = 6\ \text{Cis}\ 210°$

1) Plot Z_1 & Z_2

2) Find $Z_1 Z_2$

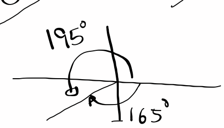
3) Find $\frac{Z_1}{Z_2}$



$$Z_1 Z_2 = 4 \cdot 6\ \text{Cis}(45° + 210°) = 24\ \text{Cis}\ 255°$$

$$\frac{Z_1}{Z_2} = \frac{4}{6}\ \text{Cis}(45° - 210°) = \frac{2}{3}\ \text{Cis}(-165°)$$

$\frac{2}{3}[\cos 165° - i \sin 165°]$



Feb 1-8:37 AM

$Z = 4 \text{ cis } 30^\circ$

find $Z^2 = Z Z$

$= (4 \text{ cis } 30^\circ)(4 \text{ cis } 30^\circ)$

$= 4^2 \text{ cis } 2 \cdot 30^\circ$

$Z^3 = Z^2 \; Z = (4^2 \text{ cis } 2 \cdot 30^\circ)(4 \text{ cis } 30^\circ)$

$= 4^3 \text{ cis } 3 \cdot 30^\circ$

$Z^6 = 4^6 \text{ cis } 6 \cdot 30^\circ$

If $Z = r \text{ cis } \theta$, then

$Z^n = r^n \text{ cis } n\theta$

Feb 1-8:48 AM

$Z = 2 \text{ cis } 45^\circ$

find $Z^3 = 2^3 \text{ cis } 3 \cdot 45^\circ$

$= 8 \text{ cis } 135^\circ$

$= 8[\cos 135^\circ + i \sin 135^\circ]$

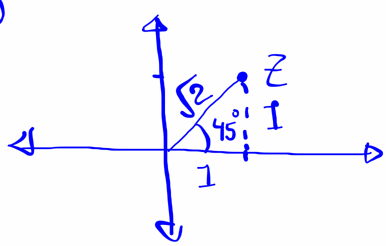
$= 8[-\cos 45^\circ + i \sin 45^\circ]$

$= 8\left[-\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right] = -4\sqrt{2} + i \cdot 4\sqrt{2}$

Feb 1-8:53 AM

find $(1+i)^{20}$

$Z = 1 + i$



$Z = \sqrt{2}\ \text{Cis}\ 45°$

$Z^{20} = (\sqrt{2})^{20}\ \text{Cis}\ 20 \cdot 45°$

$= 1024\ \text{Cis}\ 900° = 1024\ \text{Cis}\ 180°$

$= 1024[\cancel{\cos 180°}^{-1} + i\ \cancel{\sin 180°}^{0}]$

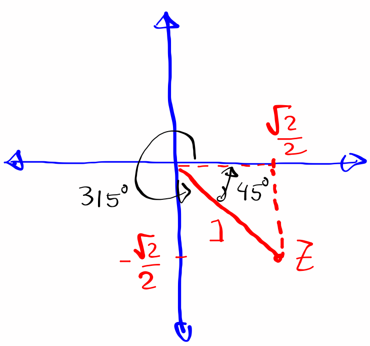
$= -1024$

Feb 1-8:56 AM

find $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)^{12}$

$Z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

$r = |Z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{-\sqrt{2}}{2}\right)^2} = 1$



$Z = r\ \text{Cis}\ \theta$

$Z = 1\ \text{Cis}\ 315°$

$3780 \div 360° = 10.5$

10 Revolution + half of Rev.

$Z^{12} = 1^{12}\ \text{Cis}\ 12(315°)$

$= \text{Cis}\ 3780°$

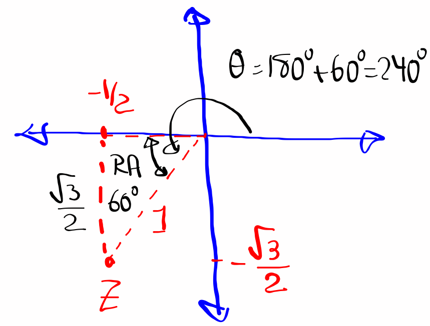
$= \text{Cis}\ 180°$

$= \cos 180° + i \sin 180°$

$= \boxed{-1}$

Feb 1-9:00 AM

Find $\left(\underbrace{-\frac{1}{2}-\frac{\sqrt{3}}{2}i}\right)^{15}$



$Z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$r = |Z| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = 1$

$Z = r \text{ Cis } \theta$

$Z = 1 \text{ Cis } 240°$

$Z^{15} = 1^{15} \text{ Cis } 15 \cdot 240°$

$= \text{Cis } 3600°$

$= \text{Cis } 0° = \cancel{\cos 0°}^{\,1} + i \cancel{\sin 0°}^{\,0}$

$Z^{15} = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{15} = \boxed{1}$

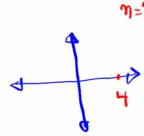
Feb 1-9:06 AM

Find the nth roots of Z.

$\sqrt[n]{Z} = \sqrt[n]{r} \text{ Cis } \frac{\theta + K \cdot 360°}{n}$ for $K = 0, 1, 2, \cdots, n-1$

Find all square roots of 4.

$Z = 4 \rightarrow Z = 4 \text{ Cis } 0°$



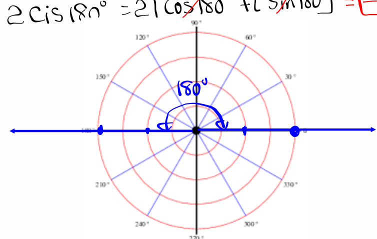
$\overset{n=2}{\sqrt{Z}} = \overset{n=2}{\sqrt{4}} \text{ Cis } \frac{0° + K \cdot 360°}{2}$

$= 2 \text{ Cis } \frac{K \cdot 360°}{2}$

$= 2 \text{ Cis } K \cdot 180°$

$K = 0 \rightarrow 2 \text{ Cis } 0° = 2\left[\cancel{\cos 0°}^{\,1} + i \cancel{\sin 0°}^{\,0}\right] = \boxed{2}$

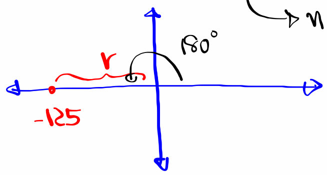
$K = 1 \rightarrow 2 \text{ Cis } 180° = 2\left[\cancel{\cos 180°}^{\,-1} + i \cancel{\sin 180°}^{\,0}\right] = \boxed{-2}$



Feb 1-9:35 AM

Find all cube roots of -125.

$\underbrace{\text{cube roots}}_{\to\, n=3}$



$n-1=2$

$-125 = 125 \text{ Cis } 180°$

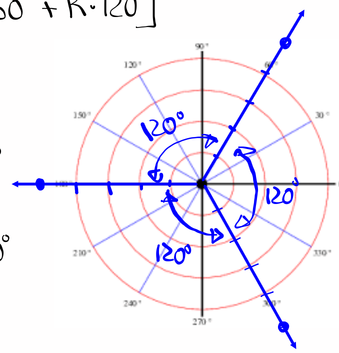
$$\sqrt[3]{z} = \sqrt[3]{125} \text{ Cis } \frac{180° + K \cdot 360°}{3}$$

$$= 5 \text{ Cis} \left[60° + K \cdot 120°\right]$$

$K=0 \Rightarrow 5 \text{ Cis } 60°$

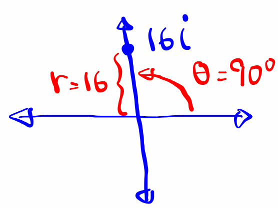
$K=1 \Rightarrow 5 \text{ Cis } 180°$

$K=2 \Rightarrow 5 \text{ Cis } 300°$



Feb 1-9:42 AM

Find all 4th roots of $16i$.



$z = 16i$

$$\sqrt[4]{z} = \sqrt[4]{16} \text{ Cis } \frac{90° + K \cdot 360°}{4}$$

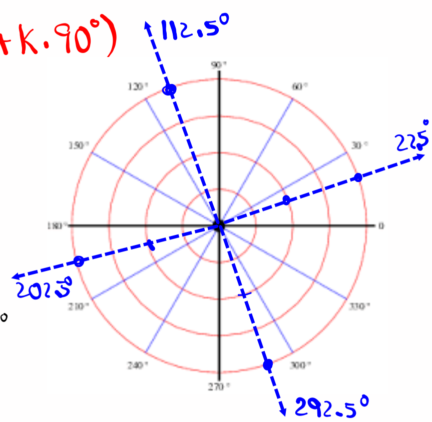
$$= 2 \text{ Cis } (22.5° + K \cdot 90°)$$

$K=0 \to 2 \text{ Cis } 22.5°$

$K=1 \to 2 \text{ Cis } 112.5°$

$K=2 \to 2 \text{ Cis } 202.5°$

$K=3 \to 2 \text{ Cis } 292.5°$



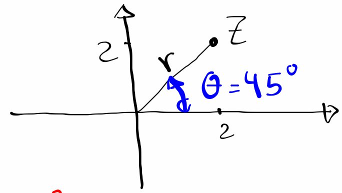
De Moivre's Theorem

Feb 1-9:50 AM

Find all cube roots of $z = 2 + 2i$

$\hookrightarrow n = 3$

$r = \sqrt{2^2 + 2^2} = \sqrt{8}$



$z = \sqrt{8}\ \text{cis}\ 45^\circ$

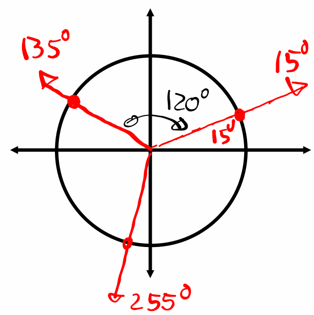
$$\sqrt[3]{z} = \sqrt[3]{\sqrt[2]{8}}\ \text{cis}\ \frac{45^\circ + k\cdot 360^\circ}{3}$$

$$= \sqrt[6]{8}\ \text{cis}\left(15^\circ + k\cdot 120^\circ\right)$$

$k = 0 \rightarrow \sqrt[6]{8}\ \text{cis}\ 15^\circ$

$k = 1 \rightarrow \sqrt[6]{8}\ \text{cis}\ 135^\circ$

$k = 2 \rightarrow \sqrt[6]{8}\ \text{cis}\ 255^\circ$



Feb 1-9:57 AM

Find all fifth roots of $-16 - 16\sqrt{3}\,i$.

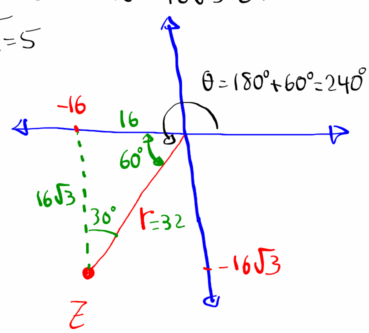
$\hookrightarrow n = 5$

$z = -16 - 16\sqrt{3}\,i$

$r = |z|$

$= \sqrt{(-16)^2 + (-16\sqrt{3})^2}$

$= 32$



$z = r\ \text{cis}\ \theta$

$= 32\ \text{cis}\ 240^\circ$

$$\sqrt[5]{z} = \sqrt[5]{32}\ \text{cis}\ \frac{240^\circ + k\cdot 360^\circ}{5} = 2\ \text{cis}\left(48^\circ + k\cdot 72^\circ\right)$$

$k = 0 \rightarrow 2\ \text{cis}\ 48^\circ$

$k = 1 \rightarrow 2\ \text{cis}\ 120^\circ$

$k = 2 \rightarrow 2\ \text{cis}\ 192^\circ$

$k = 3 \rightarrow 2\ \text{cis}\ 264^\circ$

$k = 4 \rightarrow 2\ \text{cis}\ 336^\circ$

Feb 1-10:04 AM

Solve $z^4 - 625 = 0$

4 Answers

$$(z^2 - 25)(z^2 + 25) = 0$$

$$(z - 5)(z + 5)(z^2 + 25) = 0$$

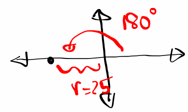
$z - 5 = 0$ $\quad$ $\boxed{z = 5}$

$z + 5 = 0$ $\quad$ $\boxed{z = -5}$

$z^2 + 25 = 0$

$z^2 = -25$

$z^2 = 25 \text{ Cis } 180^\circ$



$$z = \sqrt{25} \text{ Cis } \frac{180^\circ + k \cdot 360^\circ}{2}$$

$$= 5 \text{ Cis} (90^\circ + k \cdot 180^\circ)$$

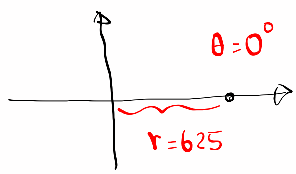
$k=0 \rightarrow 5 \text{ Cis } 90^\circ = 5[\cancelto{0}{\cos 90^\circ} + i\cancelto{1}{\sin 90^\circ}] = \boxed{5i}$

$k=1 \rightarrow 5 \text{ Cis } 270^\circ = 5[\cancelto{0}{\cos 270^\circ} + i \cdot \cancelto{-1}{\sin 270^\circ}] = \boxed{-5i}$

Feb 1-10:51 AM

Solve $z^4 - 625 = 0$

$z^4 = 625$

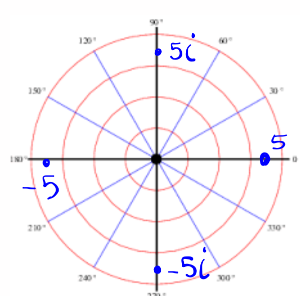


$z^4 = 625 \text{ Cis } 0^\circ$

$$z = \sqrt[4]{625} \text{ Cis } \frac{0^\circ + k \cdot 360^\circ}{4} = 5 \text{ Cis } k \cdot 90^\circ$$

$k=0 \rightarrow 5 \text{ Cis } 0^\circ = 5[\cancelto{1}{\cos 0^\circ} + i\cancelto{0}{\sin 0^\circ}] = 5$ ✓

$k=1 \rightarrow 5 \text{ Cis } 90^\circ = 5[\cancelto{0}{\cos 90^\circ} + i\cancelto{1}{\sin 90^\circ}] = 5i$ ✓



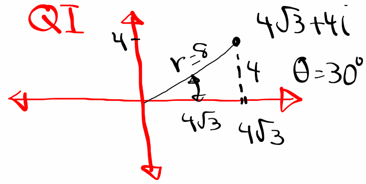
Feb 1-10:56 AM

Solve $z^3 - 4\sqrt{3} - 4i = 0$

3 Answers

$z^3 = 4\sqrt{3} + 4i$

$r = |4\sqrt{3}+4i| = \sqrt{(4\sqrt{3})^2 + 4^2} = 8$



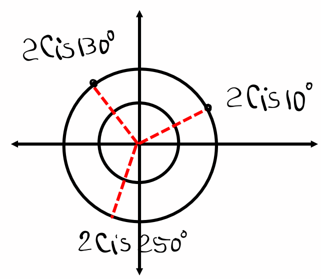
$z^3 = 8\,\text{cis}\,30°$

$z = \sqrt[3]{8}\ \text{cis}\,\frac{30° + K\cdot 360°}{3} = 2\,\text{cis}\,(10° + K\cdot 120°)$

$K=0 \Rightarrow 2\,\text{cis}\,10°$

$K=1 \Rightarrow 2\,\text{cis}\,130°$

$K=2 \Rightarrow 2\,\text{cis}\,250°$



Feb 1-11:01 AM

$z_1 = 3\,\text{cis}\,\frac{\pi}{6}$ $\qquad$ $z_2 = 5\,\text{cis}\,\frac{4\pi}{3}$

$z_1 = 3\,\text{cis}\,30°$ $\qquad$ $z_2 = 5\,\text{cis}\,240°$

Find $z_1 z_2$, and $\frac{z_1}{z_2}$

$z_1 z_2 = 3\cdot 5\ \text{cis}\,(30° + 240°) = 15\,\text{cis}\,270°$

$\frac{z_1}{z_2} = \frac{3}{5}\,\text{cis}\,(30° - 240°) = \frac{3}{5}\,\text{cis}\,(-210°)$

$= \frac{3}{5}\,\text{cis}\,(150°)$

$\frac{3}{5}\left[\cos(-210°) + i\sin(-210°)\right] =$

$\frac{3}{5}\left[\cos 210° - i\sin 210°\right]$

Feb 1-8:43 AM