Matrix is a table of numbers consists of some rows & some columns.

ex: \[
\begin{pmatrix}
1 & -4 \\
2 & 3 & 7
\end{pmatrix}
\]
2 Rows & 3 Columns
we say it is a 2x3 matrix

\[
\begin{pmatrix}
5 & 2 & -1 \\
4 & 7 & 8 \\
-3 & 10 & 2
\end{pmatrix}
\xrightarrow{\text{Rows}} 3 \times 3 \xrightarrow{\text{Square}} \xrightarrow{\text{Columns}} \text{Matrix}
\]

\[
\begin{pmatrix}
3 \\
-2 \\
5 \\
4
\end{pmatrix}
\text{Column Matrix}
\]
4x1

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1
\end{pmatrix}
\text{Row matrix.}
\]
1x5
\[
\begin{align*}
\begin{cases}
  x + y &= 5 \\
 2x - y &= 1
\end{cases}
\end{align*}
\Rightarrow
\begin{bmatrix}
  1 & 1 & 5 \\
  2 & -1 & 1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  1 & a & b \\
  0 & 1 & c
\end{bmatrix}
\]

Some Row Operations:

1) We can interchange rows
2) We can multiply or divide any row by any non-zero number.
3) We can add any multiple of any row to any other row to get a new row.

\[
\begin{align*}
\begin{array}{c}
R1 + R2 \rightarrow R2 \\
[-3] \rightarrow R2 \\
0x + 1y = 3 \rightarrow y = 3 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  x + y &= 5 \\
  x + 3 &= 5
\end{cases}
\Rightarrow
\begin{cases}
  x &= 2 \\
  \{2,3\}
\end{cases}
\end{align*}
\]

Solve by Matrix Method:

\[
\begin{align*}
\begin{cases}
  4x + 3y &= -5 \\
  x + 2y &= 0
\end{cases}
\Rightarrow
\begin{bmatrix}
  4 & 3 & -5 \\
  1 & 2 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  1 & 2 & 0 \\
  0 & 1 & 1
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  0 & 1 & 1 \\
  y = 1 \\
  \{\}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  1 & 2 & 0 \\
  0 & 1 & 1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  1 & 2 & 0 \\
  0 & 1 & 1 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  x + 2y &= 0 \\
  x + 2(1) &= 0
\end{bmatrix}
\Rightarrow
\begin{cases}
  x = -2 \\
  y = 1
\end{cases}
\Rightarrow
\{\{2,1\}\}
\end{align*}
\]

When one row becomes all Zero's, there are infinitely many solutions.

When one row becomes all Zero's except on last column, there is no Soln.
Solve by Matrix method:

\[
\begin{align*}
\begin{cases}
2x + y + z &= 0 \\
2x + y - z &= 6 \\
x - y - z &= 2
\end{cases}
\Rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
2 & 1 & -1 & 6 \\
1 & -1 & -1 & 2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & -1 & -3 & 6 \\
0 & 0 & -2 & 2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & -1 & -3 & 6 \\
0 & 0 & 4 & -10
\end{bmatrix}
\end{align*}
\]

(-2)R1 + R2 → R2
(-1)R1 + R3 → R3
R2 ÷ (-1) → R2
R3 ÷ 4 → R3

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 3 & -6 \\
0 & 0 & 1 & -2.5
\end{bmatrix}
\Rightarrow
\begin{align*}
0x + 1y + 3z &= -6 \\
x + 0y + 1z &= -2.5 \\
z &= -2.5
\end{align*}
\]

\[
\begin{align*}
x + y + z &= 0 \\
x + 1.5 - 2.5 &= 0 \Rightarrow x = 1
\end{align*}
\]

Final Ans \[\{(1, 1.5, -2.5)\}\]

Please double-check our calculations
Solve by Matrix Method:
\[
\begin{align*}
2x - y - 3z &= -9 \\
-x + y + 2z &= -5 \\
x + y - z &= 3
\end{align*}
\]

\[
\begin{bmatrix}
2 & -1 & -3 \\
-1 & 1 & 2 \\
1 & 1 & -1
\end{bmatrix}
\]

R1 → R3
\[
\begin{bmatrix}
1 & 1 & 3 \\
-1 & 2 & -5 \\
-1 & -3 & -9
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 & -1 \\
0 & 2 & 1 \\
0 & -3 & -1
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 & -1, 3 \\
0 & 2 & 1 \\
0 & -3 & -1, 6
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 & -1, 3 \\
0 & 6 & 3 \\
0 & -6 & -2, -30
\end{bmatrix}
\]

(1) R1 + R2 → R2
(2) R3 → R3
(3) R2 → R2
R2 + R3 → R3

\[
\begin{bmatrix}
1 & 1 & -1 \\
0 & 6 & 3 \\
0 & 0 & 1
\end{bmatrix}
\Rightarrow
\begin{align*}
6y + 3z &= -6 \\
2y + z &= -2 \\
-36 &= -2
\end{align*}
\]
\[
\begin{align*}
2y = 34 & \Rightarrow y = 17 \\
x + y - z &= 3 \\
x + 17 + 36 &= 3
\end{align*}
\]

x = -50

Final Soln: \{ (-50, 17, -36) \}
Solve by matrix method:
\[
\begin{align*}
2x - y + 4z &= -3 \\
x - 4z &= 5 \\
6x - y + 2z &= 10
\end{align*}
\Rightarrow
\begin{bmatrix}
2 & -1 & 4 & -3 \\
1 & 0 & -4 & 5 \\
6 & -1 & 2 & 10
\end{bmatrix}
\]
\text{R1} \rightarrow \text{R2}
\begin{bmatrix}
1 & 0 & -4 & 5 \\
2 & -1 & 4 & -3 \\
6 & -1 & 2 & 10
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 0 & -4 & 5 \\
0 & -1 & 12 & -13 \\
0 & -1 & 26 & -20
\end{bmatrix}
\text{(-2)R1} + \text{R2} \rightarrow \text{R2}
\text{(-6)R1} + \text{R3} \rightarrow \text{R3}
\begin{bmatrix}
1 & 0 & -4 & 5 \\
0 & -1 & 12 & -13 \\
0 & 0 & 14 & -7
\end{bmatrix}
\]
\(x = 3\), \(y = 7\), \(z = \frac{1}{2}\)

There is a numerical value associated with any square matrix and it is called the determinant.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Determinant</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
\text{a} & \text{b} \\
\text{c} & \text{d}
\end{bmatrix}
\] | \[
\begin{vmatrix}
\text{a} & \text{b} \\
\text{c} & \text{d}
\end{vmatrix}
\] ⇒ any numerical value it could be 0, +, or -.

1. \([\text{a}] \rightarrow |\text{a}| = \text{a} \quad \text{So} \quad |-5| = -5\)
   Not absolute value
   \(|7| = 7\)
   \(|0| = 0\)
$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$2 \times 2$

$\begin{vmatrix} 5 & 3 \\ 1 & -2 \end{vmatrix} = 5(-2) - 1(3) = -10 - 3 = \boxed{-13}$

$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 4(3) - 6(2) = 12 - 12 = \boxed{0}$

$\begin{vmatrix} 7 & 8 \\ -5 & 3 \end{vmatrix} = 7(3) - (-5)(8) = 21 + 40 = \boxed{61}$

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Expansion by first row

$= a(ei - hf) - b(di - gf) + c(dh - eg)$

$\begin{vmatrix} 3 & 2 & -4 \\ 1 & 7 & 2 \\ 4 & 9 & -2 \end{vmatrix} = 3 \begin{vmatrix} 7 & 2 \\ 9 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + (-4) \begin{vmatrix} 1 & 7 \\ 4 & 9 \end{vmatrix}$

$= 3(14 - 18) - 2(-2 - 8) - 4(9 - 28)$

$= 3(-12) - 2(-10) - 4(-19) = \boxed{0}$
Evaluate

\[
\begin{vmatrix}
3 & 0 & 1 \\
0 & -4 & 7 \\
2 & 0 & 1
\end{vmatrix}
= 5 \begin{vmatrix}
-4 & 7 \\
0 & -1
\end{vmatrix}
- 0 \begin{vmatrix}
\text{who} & +3 \\
\text{Cares!} & 2
\end{vmatrix}
+ 0 \begin{vmatrix}
0 & 4 \\
2 & 0
\end{vmatrix}
= 5(-4 - 0) + 3(0 + 8) = 20 + 24 = \boxed{44}
\]

\[
\begin{vmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{vmatrix}
= 1 \begin{vmatrix}
4 & 5 \\
0 & 6
\end{vmatrix}
- 2 \begin{vmatrix}
0 & 5 \\
0 & 6
\end{vmatrix}
+ 3 \begin{vmatrix}
0 & 4 \\
0 & 0
\end{vmatrix}
= 1(24 - 0) - 2(0 - 0) + 3(0 - 0) = \boxed{24}
\]

Solving system of equations by Cramer's Rule:

\[x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}\]

where

- \(D\) is the det. of the Coef. Matrix
- \(D_x\) is the det. of the Coef. matrix after coef. of \(x\)'s are replaced by constant numbers on the right hand side of the system
- \(D_y\)
- \(D_z\)
\[
\begin{align*}
\begin{cases}
2x - 3y &= 9 \\
5x + 4y &= 6
\end{cases}
\end{align*}
\]

\[
D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 2(4) - 5(-3) = 8 + 15 = 23
\]

\[
D_x = \begin{vmatrix} 9 & -3 \\ 6 & 4 \end{vmatrix} = 9(4) - 6(-3) = 36 + 18 = 54
\]

\[
D_y = \begin{vmatrix} 2 & 9 \\ 5 & 6 \end{vmatrix} = 2(6) - 5(9) = 12 - 45 = -33
\]

Final Ans \ \{ \left(\frac{54}{23}, \frac{-33}{23}\right) \}
\[ \begin{align*} \{ & \begin{cases} x - 3y + 7z = 13 \\ x + y + z = 1 \\ x - 2y + 3z = 4 \end{cases} \\ \{ & \begin{cases} x - y + 7z = 13 \\ x + y - z = 1 \\ x - 2y + 3z = 4 \end{cases} \end{align*} \]

Use Cramer's rule to solve for \( z \) only.

\[ D = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = -10 \]

\[ D_z = \begin{vmatrix} 1 & -3 & 13 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = -24 \]

\[ D_x = 1 \begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 13 \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 1(4+2) + 3(4-1) + 13(-2-1) = 6 + 9 - 39 = -24 \]

\[ y = \frac{D_y}{D} = \frac{1}{1} = 1 \]

\[ \begin{align*} \{ & \begin{cases} x + z = 3 \\ -x + y = 0 \\ y + 2z = 5 \end{cases} \end{align*} \]

Solve for \( y \) by Cramer's rule.

\[ D = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} \]

\[ D_y = \begin{vmatrix} 1 & 3 & 1 \\ -1 & 0 & 0 \\ 0 & 5 & 2 \end{vmatrix} \]

\[ = 1(2-0) + 1(-1-0) = 1(0-0) - 3(-2-0) + (5) = 6 - 5 = 1 \]

\[ y = \frac{D_y}{D} = \frac{1}{1} = 1 \]