Multiplication Rule in Probability with Tree Diagram

What is the **Multiplication Rule**?

Given two different events A and B, the **Multiplication Rule** states

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Where

- $P(A \text{ and } B)$ implies that event A happens first, then event B is to happen, and
- and $P(B|A)$ implies that event A has taken place already, therefore $P(B)$ has to be adjusted accordingly if necessary.

Example:

Given $P(A) = 0.6, P(B|A) = 0.4$, find $P(A \text{ and } B)$.

Solution:

Using the multiplication rule, we get

$$\begin{aligned} P(A \text{ and } B) &= P(A) \cdot P(B|A) \\ &= 0.6 \cdot 0.4 = 0.24 \end{aligned}$$

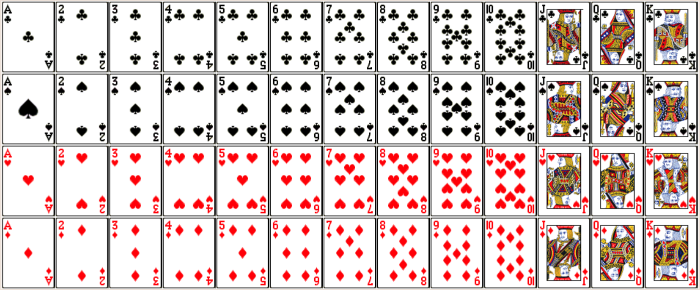
Example:

What is the probability of drawing two face cards from an ordinary full-deck of playing cards?

- With replacement
- Without replacement

Solution:

An ordinary full-deck of playing cards has 52 cards and 12 of them are face cards as shown here.



Let F_1 be the event that the first card is a face card, and F_2 be the event that the second card is a face card, using the multiplication rule, we need to evaluate the following formula.

$$P(F_1 \text{ and } F_2) = P(F_1) \cdot P(F_2|F_1)$$

Solution Continued:

Working with a full-deck of playing cards, $P(F_1) = \dfrac{12}{52}$, however when calculating $P(F_2|F_1)$, we need to know what happens to the first card.

- With replacement $\Rightarrow P(F_2|F_1) = \dfrac{12}{52}$
- Without replacement $\Rightarrow P(F_2|F_1) = \dfrac{11}{51}$

so

- With replacement
  $\Rightarrow P(F_1 \text{ and } F_2) = P(F_1) \cdot P(F_2|F_1) = \dfrac{12}{52} \cdot \dfrac{12}{52} = \dfrac{9}{169}$
- Without replacement
  $\Rightarrow P(F_1 \text{ and } F_2) = P(F_1) \cdot P(F_2|F_1) = \dfrac{12}{52} \cdot \dfrac{11}{51} = \dfrac{11}{221}$

What are **Independent Events**?

Two different events A and B, are called **Independent Events** when $P(B|A) = P(B)$ and $P(A|B) = P(A)$.

What is the property of **Independent Events**?

For any two independent events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example:

Given: $P(A) = 0.6, P(B) = 0.5$, and A and B are independent events, find $P(A \text{ and } B)$ and $P(A \text{ or } B)$.

Solution:

Since A and B are independent events, we get

$$\begin{aligned} P(A \text{ and } B) &= P(A) \cdot P(B) \\ &= 0.6 \cdot 0.5 = 0.3 \end{aligned}$$

Using the addition rule, we get

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 0.6 + 0.5 - 0.3 = 0.8 \end{aligned}$$

What are **Dependent Events**?

Two different events A and B, are called **Dependent Events** means that the probability of one event is affected by the result of an earlier event.

What is the property of **Dependent Events**?

For any two dependent events A and B,

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example:

It has been reported that in Europe, 88% of all households have a TV. 55% of all households have a TV and a DVR. What is the probability that a randomly selected household in Europe has a DVR given that it has a TV?

Solution:

Let T be the event that a randomly selected household has a TV and D be the event that a randomly selected household has a DVR. So $P(T) = 0.88$ and $P(T \text{ and } D) = 0.55$, now we can use the conditional probability formula to find $P(D|T)$.

$$P(D|T) = \frac{P(T \text{ and } D)}{P(T)} = \frac{0.55}{0.88} = 0.625$$

Example:

The probability that a freshman taking a math class is 0.75. The probability of taking a math class and an English class is 0.4. What is the probability that a randomly selected freshman taking an English class given that he or she is taking a math class?

Solution:

Let M be the event that a randomly selected freshman is taking a math class and E be the event that a randomly selected freshman is taking an English class. So $P(M) = 0.75$ and $P(M \text{ and } E) = 0.4$, now we can use the conditional probability formula to find $P(E|M)$.

$$\begin{aligned} P(E|M) &= \frac{P(M \text{ and } E)}{P(M)} \\ &= \frac{0.4}{0.75} = 0.533 \end{aligned}$$

What is a **Tree Diagram**?

Tree diagrams are a helpful tool for calculating probabilities when there are several events involved.

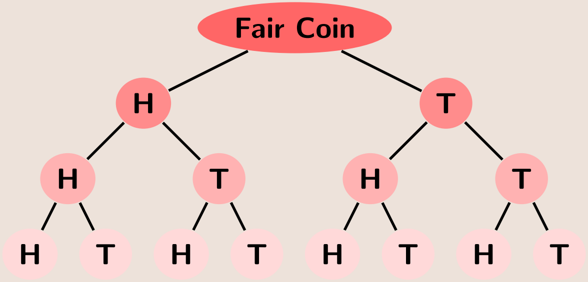
How do we use the **Tree Diagram**?

- The probability of each branch is written on the branch.
- The outcome is written at the end of the branch.
- Multiply probabilities along the branches.
- Add probabilities of those branches that satisfy the desired event.

Example:

Use tree diagram to display all outcomes when tossing a fair coin three times.

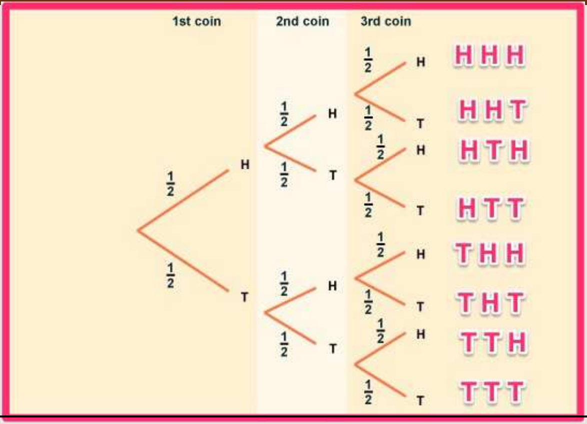
Solution:



Example:

Use tree diagram to display all outcomes with indicated probabilities for each branch when you toss three fair coins.

Solution:



Example:

Suppose a fair coin is tossed three times,

- find the probability of getting all tails.
- find the probability of getting all heads.
- find the probability of getting all tails or all heads .
- find the probability of getting neither all tails nor all heads .

Solution:

Using the tree diagram, we get

$$P(\text{ All Tails }) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Solution Continued:

Similarly,

$$P(\text{ All Heads }) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

then,

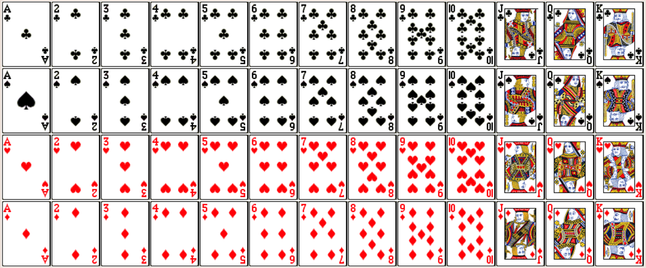
$$\begin{aligned} P(\text{All Heads or All Tails}) &= P(\text{All Heads}) + P(\text{All Tails}) \\ &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \end{aligned}$$

and,

$$\begin{aligned} P(\overline{\text{All Heads or All Tails}}) &= 1 - P(\text{All Heads or All Tails}) \\ &= 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

Example:

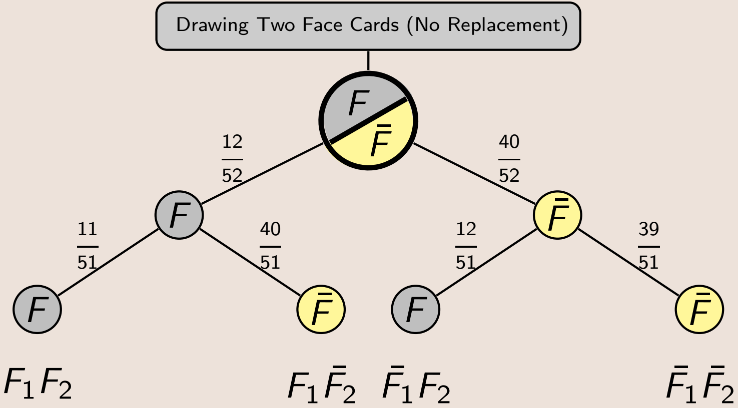
An ordinary full-deck of playing cards has 52 cards and 12 of them are face cards as shown below.



Use the tree diagram to find the probability of all outcomes of selecting two face cards without replacement.

Solution:

Let F_1 be the event that the first card is a face card, and F_2 be the event that the second card is a face card, and similarly Let $\bar{F}_1$ be the event that the first card is not a face card, and $\bar{F}_2$ be the event that the second card is not a face card.



Solution Continued:

$$P(F_1 \text{ and } F_2) = \frac{12}{52} \cdot \frac{11}{51} = \frac{11}{221}$$

$$P(F_1 \text{ and } \bar{F_2}) = \frac{12}{52} \cdot \frac{40}{51} = \frac{40}{221}$$

$$P(\bar{F_1} \text{ and } F_2) = \frac{40}{52} \cdot \frac{12}{51} = \frac{40}{221}$$

$$P(\bar{F_1} \text{ and } \bar{F_2}) = \frac{40}{52} \cdot \frac{39}{51} = \frac{130}{221}$$

It is worth mentioning that if we add all these probabilities, we do get 1 as expected.